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United States  
Department of  
Agriculture

Forest Service

Rocky Mountain  
Forest and Range  
Experiment Station

Fort Collins,  
Colorado 80526

General Technical  
Report RM-173



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# Forest Service Land Management Planners' Introduction to Linear Programming

Brian M. Kent



# Forest Service Land Management Planners' Introduction to Linear Programming

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## Abstract

This report provides a detailed explanation of basic concepts of linear programming (LP) for forest planners and others who must use the technique to develop national forest land management plans. The role of LP in national forest planning analysis is presented. Basic algebraic concepts needed to understand LP are reviewed. The various components of an LP model are described and the advantages and limitations of the technique are discussed. Examples are presented in order to illustrate the technique, how it is used to formulate forest planning models, and how it is used in the Forest Service land management planning process. Procedures for solving LP models and the technique known as goal programming are also discussed. No prior background in LP is assumed on the part of the reader.

<sup>1</sup>Headquarters is in Fort Collins, in cooperation with Colorado State University.

## Contents

	Page
INTRODUCTION .....	1
A FOREST SERVICE SYSTEMS ANALYSIS OR OPERATIONS	
RESEARCH APPROACH TO MULTIPLE USE PLANNING .....	1
SYSTEMS ANALYSIS IN NATURAL RESOURCE MANAGEMENT ..	2
LINEAR PROGRAMMING AND THE PLANNING PROCESS .....	2
COMPUTER PROGRAMS FOR SOLVING LP PROBLEMS .....	4
REVIEW OF ELEMENTARY GRAPHICS AND PLOTTING (COORDINATE	
SYSTEMS) .....	5
COORDINATE SYSTEMS—AN EXPLANATION .....	5
Plotting Single Points Corresponding to Constant Values .....	5
Plotting Points Corresponding to Constant Values for $x_1$ or $x_2$ .....	5
INEQUALITIES: $x_1$ AND/OR $x_2$ ARE NOT CONSTANT VALUES ...	6
Plotting Inequality Relationships Involving a Single Variable .....	7
LINEAR RELATIONSHIPS INVOLVING BOTH $x_1$ AND $x_2$ .....	8
Plotting Equations Involving Both $x_1$ and $x_2$ .....	8
Plotting Inequalities Involving Both $x_1$ and $x_2$ .....	9
INTRODUCTION TO LINEAR PROGRAMMING .....	10
STOCKING A RANCH EXAMPLE .....	10
Constraints (Expressed Mathematically) .....	10
GENERAL REMARKS ABOUT LINEAR PROGRAMMING .....	11
THE GENERAL LINEAR PROGRAMMING MODEL .....	11
THE RANCH MODEL .....	11
OBJECTIVE FUNCTION—QUANTIFIES THE CRITERION OF	
OPTIMALITY .....	12
CONSTRAINTS .....	12
DECISION (ACTIVITY) VARIABLES .....	13
DEFINITIONS .....	13
MORE LINEAR PROGRAMMING TERMINOLOGY—MATRIX AND	
VECTOR .....	13
Case 1.—The Vector of Objective Function Coefficients .....	14
Case 2.—The Vector of Decision Variables .....	14
Case 3.—The Matrix of Coefficients of Activity Variables on the Left	
Hand Sides of the Constraints .....	14
Case 4.—The Vector of Right Hand Sides for the Constraints .....	14
THE MATRIX APPROACH TO EXPRESSING THE LP MODEL .....	14
THE PROBLEM OF UNITS .....	15
A GRAPHICAL INTERPRETATION OF THE “STOCKING A RANCH”	
PROBLEM .....	16
DETERMINING THE SET OF FEASIBLE SOLUTIONS .....	17
Wife's Sheep Constraint .....	17
Ranch Hands' Constraint .....	17
Rancher's Constraint .....	17
Forage Constraint .....	18
DETERMINATION OF THE OPTIMAL SOLUTION .....	18
Determining the Objective Function .....	18
THE SIMPLEX METHOD .....	20
Slack Variables .....	20
Surplus Variable .....	21
A LAND USE PROBLEM AND SOME ELEMENTARY SENSITIVITY	
ANALYSIS .....	21
A SIMPLE ALLOCATION PROBLEM: WILSON'S TIMBER-	
RECREATION MODEL .....	21
WILSON'S TR MODEL AND THE NFMA PLANNING PROCESS ...	22
Management Practices and Management Prescriptions .....	22
Analysis Areas .....	23
Limitations of an LP Allocation Model .....	23
The NFMA Planning Process .....	23



GENERATING ALTERNATIVE PLANS BY CHANGING THE MODEL	24
Changing the Model to Perform a Sensitivity Analysis	24
Altering the Objective Function—Maximize Timber Production	25
Altering the Objective Function—Minimize Budget	26
SOME ADDITIONAL ASPECTS OF LP PROBLEM FORMULATION	26
Infinite Number of Optimal Solutions	26
Redundant Constraints	27
No Feasible Solution	27
Unbounded Optimal Solution	28
ADVANTAGES AND DISADVANTAGES OF LINEAR PROGRAMMING	29
ASSUMPTIONS OF LINEAR PROGRAMMING	29
Linearity	29
Proportionality	29
Additivity	30
Non-Negativity	30
Divisibility	30
Certainty	30
A Single Objective Function	30
ADVANTAGES AND DISADVANTAGES OF LINEAR PROGRAMMING	30
The Linearity Assumption: A Discussion	31
The Divisibility Requirement: A Discussion	31
LP as a Deterministic Model: A Discussion	31
Nonquantifiable Benefits: A Discussion	32
MODEL SIZE	32
GOAL PROGRAMMING: A SPECIAL CASE OF LINEAR PROGRAMMING	32
An Example	33
Defining the First Goal	33
Defining the Second Goal	33
Dealing with Incommensurable Objectives	34
LITERATURE CITED	35

# Forest Service Land Management Planners' Introduction to Linear Programming

Brian M. Kent

## INTRODUCTION

The purpose of this training report is to provide a detailed explanation of the aspects of linear programming that must be understood if forest planners and other forest personnel are going to use the tool effectively in the development of national forest land use plans. The choice of topics<sup>2</sup> covered is directed to this end.

Linear programming (LP) is a member of a group of techniques or models known as mathematical programming techniques. Other members of this group include goal programming, which is a special case of LP, integer programming, and a variety of nonlinear programming models such as quadratic and geometric programming. All of these techniques are classified as optimization models in that they are designed to choose an "optimal" alternative from a set of possible or feasible alternatives. The optimal alternative is the one which yields the minimum or maximum value of some numerically measurable criterion of performance.

Each of these models is composed of a set of mathematical relationships which are functions of activities that comprise the alternatives. These relationships describe the criterion of optimality (objective function) and the set of feasible alternatives (constraints due to limitations of the system being modeled). In the linear programming model, all mathematical relationships are linear. In nonlinear programming models, one or more of the relationships are nonlinear.

Mathematical programming models are members of a larger group of techniques classified as operations research techniques. A precise definition of the term "operations research," and one that would be agreeable to all people, is difficult to state. One possible definition is that operations research techniques incorporate a scientific methodology for making decisions. Computer simulation, queuing theory, decision theory, and network analysis are other examples of operations research techniques.

Like operations research, the term "systems analysis" means different things to different people. Often the two terms are regarded as synonyms and that is the approach taken here.

The application of operations research techniques to the solution of real world problems first occurred during World War II in the areas of military planning and logistics. Since that time, applications have been

<sup>2</sup>Certain important aspects of LP, most notably the dual problem and certain refined solution techniques, have been omitted. These topics are covered in one or more of several standard LP texts. The partial list of these may be found at the end of this text.

developed in such disciplines as economics, statistics, and the social sciences.

For example, one important preliminary step in the implementation of any major project is development of a timetable, or schedule of activities, that must be completed to finish the project. A network analysis technique known as PERT (Program Evaluation and Review Technique) is often used in the development of such a timetable. The technique is also useful in helping to identify which activities or steps are critical in the sense that they must be completed on time for the overall project schedule to be met. These activities define what is known as the critical path.

The development of high-speed computers has been of fundamental importance to increasing application of operations research techniques to large scale real world problems. Consider linear programming, the operations research technique to which this report is devoted: The algorithm used to solve an LP problem is known as the simplex method. Because of the amount of computation, it would take several hours to manually solve a problem involving even three or four variables. Problems arising from development of multiple-use land management plans may involve hundreds or thousands of variables. Solution of such large problems by hand is clearly impossible. The only practical approach is use of a high-speed computer and the sophisticated solution algorithms on such a machine.

Note that "program" has different meanings. There are "mathematical programs" and "computer programs." In the former term, "program" can be regarded as a synonym for "planning," while in the latter case, "program" refers to a set of instructions to a computer written in a special computer programming language such as Fortran. Confusion on this dual usage of program is compounded by the fact that mathematical programming problems are solved with the aid of computer programs.

## A FOREST SERVICE SYSTEMS ANALYSIS OR OPERATIONS RESEARCH APPROACH TO MULTIPLE USE PLANNING

One major problem facing the manager of wildlands is satisfying the growing demand of the American people for various products and experiences such lands can provide. Public lands, including the national forests, are being called on to satisfy a large proportion of these demands. The diversity of demands leads to conflict, one which increases both the difficulty and number of problems a manager faces. For example, certain interest



groups favor a preservationist approach to land management that involves the designation of large areas of land as wilderness or national parks. These areas would be managed for the preservation of natural flora and fauna, while the production of most commodities would be deemphasized. On the other hand, other interest groups favor land management policies that enhance the production of commodities such as timber and forage (Kent 1980).

As managers address the problem of how to best satisfy these demands, they must determine the productive capabilities of the resources. This involves determining (1) the suitability of the area for producing the desired products/experiences and (2) the level or rate at which such products/experiences may be produced without destroying long-term productive potential. Unfortunately, determination of productive capabilities is complicated by poor understanding of the biological systems that comprise even a small wildland resource. As a result, impacts—especially long-term impacts—of some management strategies can be difficult to estimate. As an extreme example of this problem, some clearcuts have been conducted on poor site lands and regeneration has proved difficult, with the result that sites are unstocked for years after harvesting.

## **SYSTEMS ANALYSIS IN NATURAL RESOURCE MANAGEMENT**

Because of problems like the ones discussed above, the development of a comprehensive, workable, and effective land management plan for a large wildland system such as a national forest is a very complex process. This complexity is reflected in the planning process outlined in Section 219.5, Regional and Forest Planning Process contained in the National Forest Management Act planning regulations.

It will be necessary to consider a large number of issues, demands, management strategies, and system responses to management. A great deal of data must be assimilated and a rational procedure for development and evaluation of alternative plans designed. Such plans must be developed in a way that ensures responsiveness to public issues and management concerns; utilization of the best technical information available; and most importantly, responsiveness to changes in the public issues, management concerns, and technical information (DeVilbiss and Lundeen 1978).

All these factors combine to create a situation where effective management is both important and difficult to accomplish.

Situations such as these lend themselves to the application of systems analysis techniques. While it is possible to develop effective land management plans without the aid of these techniques, experience of the Systems Application Unit for Land Management Planning (SAU-LMP) group and others has shown that such techniques are extremely useful in plan development. Linear programming (LP), in particular, has proved to be a useful analytical aid in this process. For a general discussion

of how LP and other analytical aids have been used by the SAU-LMP group, the reader is referred to the guide developed by Betters (1977). "Linear Programming and the Planning Process" in this section describes one scheme for incorporating LP into the process of developing a forest plan.

While this report concentrates on application of LP to land management planning, it is important to realize that this technique has been applied to a wide variety of problems in forest or natural resource management. A search of the literature by Bare in 1971 produced 336 references containing applications of systems analysis techniques (Bare 1971). In 1973, Martin and Sendak published an annotated bibliography of 416 references of such applications (Martin and Sendak 1973), and as of 1976, there were 925 applications of these techniques to forest land management problems cited in the literature (Bare and Schreuder 1976).

This increase is a reflection of the effectiveness of these tools as aids to forest land management. The majority of these applications incorporate either LP, computer simulation techniques, or some combination of both. Reasons for the predominance of the use of LP techniques over other mathematical programming techniques are discussed in the Advantages and Disadvantages of Linear Programming section. As the need for such management increases, the importance of systems analysis techniques and management aids should also increase.

## **LINEAR PROGRAMMING AND THE PLANNING PROCESS**

Section 219.5 of the NFMA planning regulations specifies the 10 minimum interrelated actions that comprise the planning process:

1. Identification of issues, concerns, and opportunities;
2. Development of planning criteria (general and evaluation criteria);
3. Inventory data and information collection;
4. Analysis of the management situation;
5. Formulation of alternatives;
6. Estimated effects of alternatives;
7. Evaluation of alternatives;
8. Selection of alternative;
9. Plan implementation; and
10. Monitoring and evaluation.

It is important to note that while some of these actions are sequential with respect to time, some occur at the same time and others are repeated throughout the planning process.

The above list of actions is certainly not the only possible approach utilized to develop a forest plan; it is, however, representative of the type of planning process required. Implementation requires a large number of diverse types of activities conducted by people possessing different expertise. For example, the work plan for the development of the Arapaho-Roosevelt National Forest Plan indicates at least 250 activities are required



to implement the NFMA process (DeVilbiss and Lundeen 1978).

Because of the need for different kinds of expertise, the formation of an effective interdisciplinary (ID) team is one of the most important steps in this process.

Members of this team must have a general understanding of the role of LP in the planning process, even though they may not be directly involved with formulation and solution of the model. As Betters (1977) has pointed out, LP is only one of a number of analytical aids available for use in plan development. This would suggest, and correctly so, that formulation and solution of a linear program is only one aspect of the planning process. Unfortunately, there is a tendency to believe that "LP" and "planning" are synonymous. The formulation and solution of a linear program is often considered to be the same as the actual development of the plan. In reality, it is only one step, albeit an important one, in the overall planning process.

One way to clarify the role of LP in the planning process is by way of an example. A list of some principal planning activities and their association with the 10 major planning actions identified by the NFMA regulations is given in table 1 (adapted from DeVilbiss and Lundeen 1978). Activities directly associated with the linear programming phase of plan development are identified.

With the exception of "revise and amend forest plan," the LP-related activities occur in the middle of the planning process (see table 1). Consideration of the types of activities that come at the beginning and end of this process can shed light on the role of LP.

Activities associated with the first two NFMA actions are concerned with laying the groundwork for the plan. They are directed toward determination of the plan composition in that they identify, among other things, issues to be addressed, demands made on the forest, and management concerns. Results of these activities are major factors in the determination of the basic form the plan will take.

The activities that follow the "Estimated Effects of Alternatives" deal with selection, approval, and implementation of the preferred plan. This suggests that linear programming-related activities bridge the gap between determination of plan design, and selection and implementation of a preferred plan. In order to accomplish this, LP fulfills the following functions:

**1. Information synthesis.**—A representative model of any system as large and complex as a national forest must be capable of utilizing large quantities of information or data. For example, such data include commodity production rates or resource output coefficients for different management prescriptions. Linear programming synthesizes the data obtained by earlier activities in the "Inventory Data and Information Collection" segment of the planning process. (Material in quotes indicate separate activities discussed in table 1.) This information is utilized along with the results of the "Capability" and "Analysis area delineation" activities in formulation of the basic model. Feasible management activities as determined by the "Management prescription development" activity and "Resource output coef-

Table 1.—Some planning activities and their relationship with the NFMA planning actions.

<u>Identification of Issues, Concerns and Opportunities</u>
Public involvement
Issue strategies
Futures
Listing of issues
Management concerns
Interagency coordination
RPA/regional plan
Legal review (ongoing)
Research (ongoing)
<u>Development of Planning Criteria</u>
Process criteria
Decision criteria
<u>Inventory Data and Information Collection</u>
Resource inventory
Information systems design
Land stratification system <sup>1</sup>
Interagency coordination
<u>Analysis of the Management Situation</u>
Capability area delineation <sup>1</sup>
Resource output coefficients <sup>1</sup>
Suitability/capability analysis
Analysis area delineation <sup>1</sup>
Management prescription development <sup>1</sup>
Demand analysis
Model design <sup>1</sup>
LP matrix generation <sup>1</sup>
Supply analysis <sup>1</sup>
Projection of current management direction <sup>1</sup>
Alternative formulation <sup>1</sup>
<u>Formulation of Alternatives</u>
Public involvement
Alternative formulation <sup>1</sup>
<u>Estimated Effects of Alternatives</u>
Alternative analysis <sup>1</sup>
Cost-benefit analysis
Input-output analysis
Environmental impact analysis <sup>1</sup>
<u>Evaluation of Alternatives</u>
Comparative analysis of alternatives
Recommendation of preferred alternative
Draft environmental statement
Draft forest plan
<u>Selection of Alternative</u>
Public involvement
Recommendation of preferred alternative
Final environmental statement
Final forest plan
<u>Plan Implementation</u>
<u>Monitoring and Evaluation</u>
Monitor forest plan
Revise and amend forest plan <sup>1</sup>

<sup>1</sup>Activity directly associated with linear programming.

cients" as determined from the "Resource inventory" and "Suitability/capability" activities are incorporated. The "LP matrix generation" activity deals both with coefficient generation and with structuring and storing the model in the computer.

**2. Allocation of resources.**—Once the model is formulated, it may be solved using one of the computer software packages specifically designed for this purpose. (More on this later.) The solution consists of an allocation of resources [i.e., the number of acres managed by each management prescription and the quantity of products (timber, forage, water, sediment, etc.) produced]. Because of this, the LP model is often referred to as an “allocation model.” Linear programming models may also be used to schedule management prescriptions and resource outputs over time. The solution provides the basis for a plan, although the plan is comprised of more than just this solution (see the Advantages and Disadvantages of Linear Programming section). Correct LP model formulation ensures that the solution will be consistent with the ecological capabilities of the land base. This is done, in part, by incorporating ecological limitations in the model in the form of constraints.

**3. Alternative plan analysis.**—Linear programming is a very useful aid in formulating and quantifying the outputs from each alternative plan considered, including the so-called “no action” or projection of the current management direction alternative. Since these results provide the basis for both the analysis of estimated effects and evaluation of tradeoffs between different alternatives, this step is vital to the development of a preferred alternative. Each time the model is reformulated or modified in any way and solved again, an alternative plan is generated. Examples of how this is done are presented in subsequent sections of this report. Operations researchers refer to these types of activities as sensitivity or post-optimality analysis. In the same way, LP may aid in revision of an existing forest plan so that it may respond to changing conditions.

While this discussion presents a simplified representation of the planning process, it shows how an LP model can be utilized to help formulate a land management plan based on information obtained in the early phases of the planning process. Since LP is one of the few techniques that can be used to organize and synthesize large amounts of information required to develop a land and resource management plan, it has proven to be an effective aid in past efforts. It is important to remember that both reliable data and correct model formulation are required.

## COMPUTER PROGRAMS FOR SOLVING LP PROBLEMS

Mention has been made of computer software packages that can solve linear programming problems. A number of these are available at the Fort Collins Computer Center. These may be grouped into two categories. The first consists of packages that actually solve an LP problem using a modified form of the simplex method (see the Graphical Interpretation of the “Stocking a Ranch” Problem section for a brief introduction to the simplex method). Two packages, ILONA and FMPS (Functional Mathematical Programming System), are available but FMPS is used in most cases.

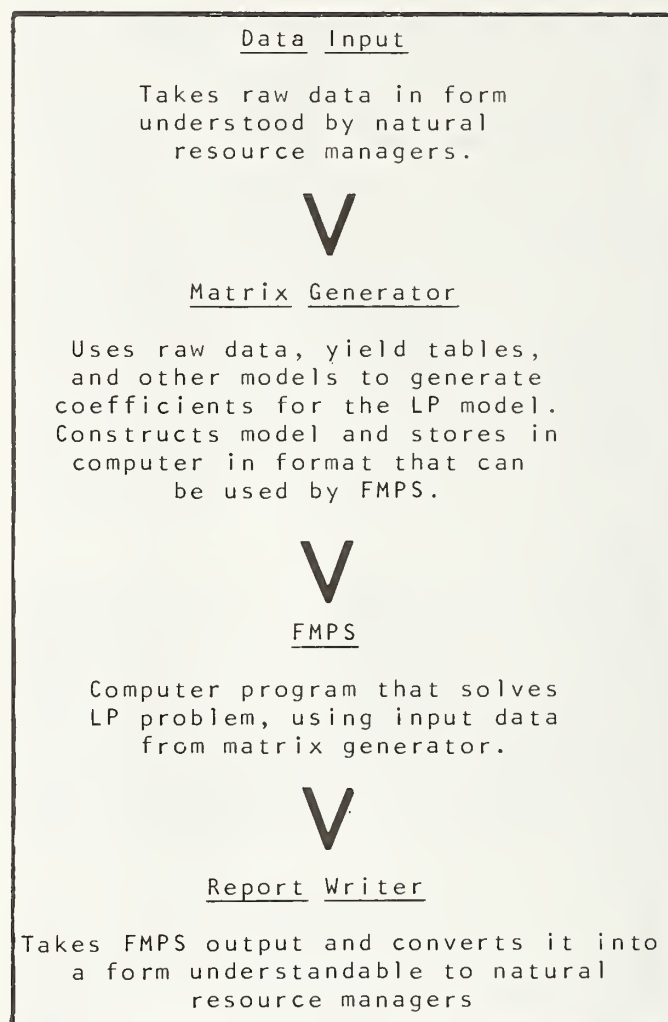


Figure 1.—Components of a typical LP package (including the FMPS solution package).

The second category is comprised of those packages that utilize user input data to structure the LP problem and then print out the results after the problem has been solved. A simplified diagram of the steps involved and the relationship between them and the solution package (in this case FMPS) is shown in figure 1. Forest Service packages in this second category include RCS (Resource Capability System), RAA (Resource Allocation Analysis), and FORPLAN (FORest PLANning model) for multiple use land management planning. Timber RAM (Resource Allocation Method) and MUSYC (Multiple Use Sustained Yield Calculation package) are Forest Service timber harvest scheduling packages. Timber RAM is a Model 1 formulation while MUSYC contains both Model 1 and Model 2. Roading RAM is an extension of Timber RAM that uses mixed-integer programming to incorporate road construction directly in the model. MAGE 5 is the matrix generator used in the RCS and RAA packages. Note that all of these packages use FMPS to solve the LP problem under consideration.

The remainder of this report is devoted to an elementary detailed coverage of LP. Additional insight into the role of linear programming in the planning process will be gained once the technique is understood.



## REVIEW OF ELEMENTARY GRAPHICS AND PLOTTING (COORDINATE SYSTEMS)

In order to understand even a simple linear programming (LP) problem, certain fundamental concepts about coordinate systems and plotting of equations and inequalities must be understood. (This section of the report may be either skimmed or skipped if you are comfortable with this material. Most readers will probably find that they have been exposed to many of these concepts, but will find a review helpful.)

### COORDINATE SYSTEMS—AN EXPLANATION

You are probably familiar with the concept of an x-y coordinate system. Such a system is used to assign an address, in the form of an x or a y coordinate value, to each point in a plane containing both an x and y axis. (Coordinate system is shown in fig. 2.)

Since it does not matter what the axes are named, we will change the name of the x axis to the  $x_1$  axis and the y axis to the  $x_2$  axis. This change is made to conform with standard linear programming notation (introduced later on). Each point on the plane defined by this coordinate system may be located or described by assigning values  $x_1$  and  $x_2$ . The two axes divide the plane into four sections or quadrants. Each quadrant is characterized by the sign (negative or positive) of the values of  $x_1$  and  $x_2$  that corresponds to points in that quadrant. These signs may be determined by examining the signs of the values of  $x_1$  and  $x_2$  found on the portions of the axes bordering the quadrant.

For example, the first quadrant is bordered by the portion of the  $x_1$  axis that has positive values for  $x_1$  and by

the portion of the  $x_2$  axis that has positive value for  $x_2$ . Thus, all points in the first quadrant are characterized by the fact that they correspond to positive values for both  $x_1$  and  $x_2$ . The values of  $x_1$  and  $x_2$  assigned to any point are called the coordinates of that point.

Points in the remaining three quadrants may be characterized in the same manner:

Quadrant	Sign of coordinate $x_1$	Sign of coordinate $x_2$
1	+	+
2	+	-
3	-	-
4	-	+

### Plotting Single Points Corresponding to Constant Values

The point where the  $x_1$  and  $x_2$  axes intersect is called the origin and has coordinate values of  $x_1 = 0$  and  $x_2 = 0$ . To find coordinates for any other point, draw a vertical line from the point to the  $x_1$  axis and read off the value of  $x_1$  at the point where the two lines intersect. Draw a horizontal line from the point to the  $x_2$  axis and read off the value of  $x_2$  at the point where the two lines intersect. Examples of this procedure are displayed in figure 3.

### Plotting Points Corresponding to Constant Values for $x_1$ or $x_2$

Until now we have talked only about locating single points on a coordinate system. Let us consider the location of several points at the same time. As an example,

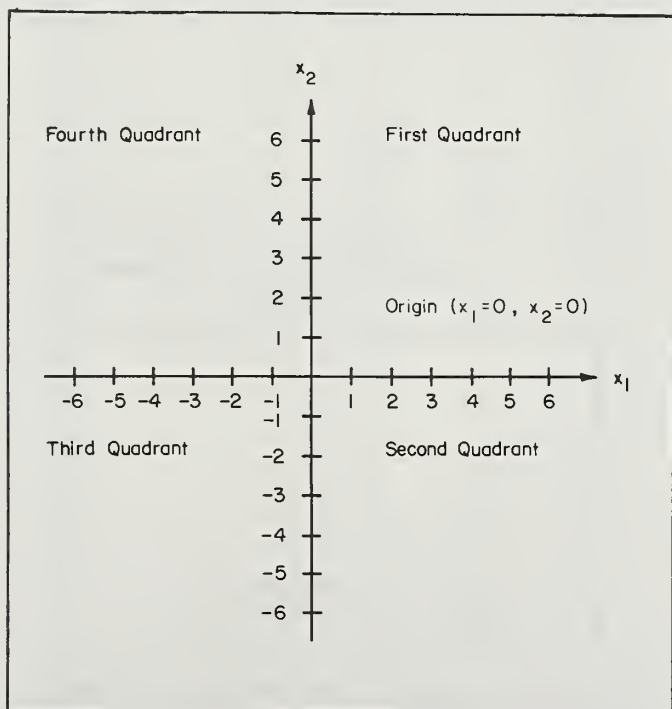


Figure 2.—An  $x_1, x_2$  coordinate system.

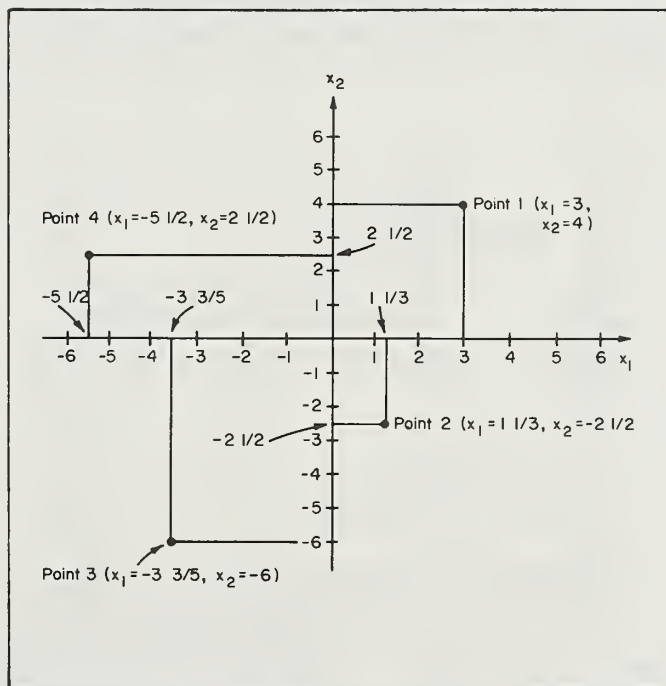


Figure 3.—Example of locating coordinates for four points on an  $x_1, x_2$  coordinate system.



how would we locate all points which have an  $x_1$  coordinate equal to 4? Before considering an answer to this question, we will simplify matters by restricting our consideration to the first quadrant only. The reason for this follows:

In an LP context,  $x_1$  and  $x_2$  are called *activity variables* or *decision variables* (more on this later). In any given linear programming problem, these variables have some physical interpretation such as the amount of some commodity or the number of acres on which particular management strategies are applied. For example, we might define them in this way:

- $x_1$  = the number of acres on which spraying and seeding of rangeland is conducted; and
- $x_2$  = the number of acres on which timber is harvested.

Clearly, it makes no sense to talk about a negative number of acres or a negative amount of some commodity. Since this will be the case for all activity variables associated with a land use planning linear program, we can ignore all quadrants except the first. In other words, we will never need to consider negative values of  $x_1$  and  $x_2$ .

Now we will return to the question of how we might plot a coordinate system where the points have an  $x_1$  coordinate of 4. Since we are only interested in the first quadrant, let us add the restriction that  $x_2$  must be positive. A plot of the line representing all points that satisfy these restrictions is displayed in figure 4. Note that it consists of a vertical line intersecting the  $x_1$  axis at the point  $x_1 = 4$ .

This general idea would hold for the plot of the set of points satisfying the requirements that  $x_1 = c$ , where  $c$  is any positive constant, and  $x_2$  is positive. That is, the

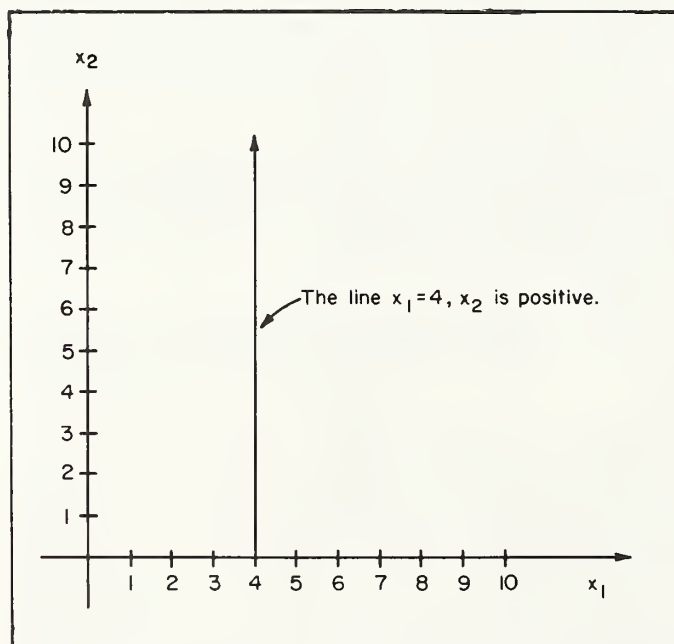


Figure 4.—Plot of a portion of the line segment representing the points that satisfy the requirements that  $x_1 = 4$  and  $x_2$  is positive.

plot would be represented by a vertical line starting on the  $x_1$  axis at the point  $x_1 = c$ .

Let us now consider plotting all the points that satisfy the requirements that  $x_2 = d$ , a positive constant, and  $x_1$  is positive. By the way of specific example, consider the points satisfying  $x_2 = 6$  and  $x_1$  is positive. This plot is shown in figure 5. Note that it consists of a horizontal line intersecting the  $x_2$  axis at the point  $x_2 = 6$ . For the arbitrary constant  $d$ , the corresponding plot would be a horizontal line intercepting the  $x_2$  axis at the point  $x_2 = d$ .

**Example 1.**—Plot the points satisfying the following set of requirements:

- $x_1 = 7$ ,  $x_2$  positive;
- $x_1 = 5 \frac{1}{2}$ ,  $x_2$  positive;
- $x_2 = 2$ ,  $x_1$  positive;
- $x_2 = 9$ ,  $x_1$  positive.

The plots of these requirements are presented in figure 6.

### INEQUALITIES: $x_1$ AND/OR $x_2$ ARE NOT CONSTANT VALUES

The material previously discussed deals with the plotting of relationships that specify that  $x_1$  or  $x_2$  are equal to some constant. Such a relationship is called an equality. Suppose that we wish to generalize this idea to consider cases where  $x_1$  and/or  $x_2$  are not restricted to being equal to some constant.

For example, consider  $x_1$  and an arbitrary constant  $c$ . Two cases are possible:

1.  $x_1$  is less than or equal to  $c$ ;
2.  $x_1$  is greater than or equal to  $c$ .

This type of relationship is known as an *inequality* and

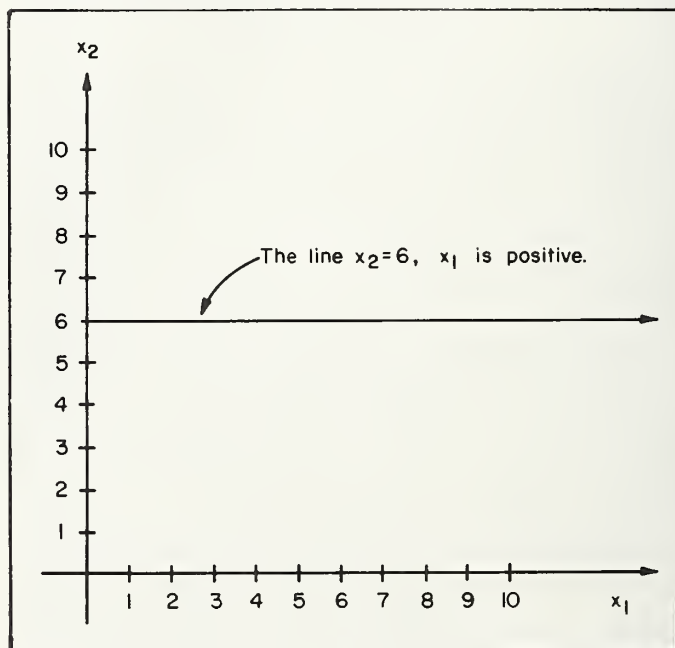


Figure 5.—Plot of a portion of the line segment representing the points that satisfy the requirement that  $x_2 = 6$  and  $x_1$  is positive.

it can be expressed mathematically with the aid of inequality symbols. These symbols are written as  $\leq$  and  $\geq$ .

For the two cases listed above:

1.  $x_1 \leq c$  ( $x_1$  is less than or equal to  $c$ );
2.  $x_1 \geq c$  ( $x_1$  is greater than or equal to  $c$ ).

Case 1 can also be stated in the form  $c$  is greater than or equal to  $x_1$ , and case 2 can also be stated in the form  $c$  is less than or equal to  $x_1$ .

The previous discussion pertains to the case which permits the two quantities  $x_1$  and  $c$  to be equal. The symbols  $>$  and  $<$  are used to express relationships where the quantities  $x_1$  and  $c$  can never equal each other. For example, the expression  $x_1 < c$  means that  $x_1$  is always less than  $c$ . Such relationships are called strict inequalities.

### Plotting Inequality Relationships Involving a Single Variable

We have defined inequalities of the following forms:

- $$\begin{aligned} x_1 &\leq c \\ x_1 &\geq c \\ x_2 &\leq d \\ x_2 &\geq d \end{aligned}$$
- where  $c$  and  $d$  are arbitrary constants.

Let us consider how we might plot the set of points that satisfy these relationships. To see how this is done, we will first consider a specific example with  $c = 4$  and  $d = 6$ . That is, we are interested in the inequalities  $x_1 \leq 4$ ,  $x_1 \geq 4$ ,  $x_2 \leq 6$ , and  $x_2 \geq 6$ . As a first step, consider plotting the equalities  $x_1 = 4$  and  $x_2 = 6$  as we did above.

The line corresponding to  $x_1 = 4$  is plotted in figure 7. Note that, as before, we are only interested in non-negative values of  $x_1$  and  $x_2$  so that only the first quadrant has been plotted. Also note that, as indicated

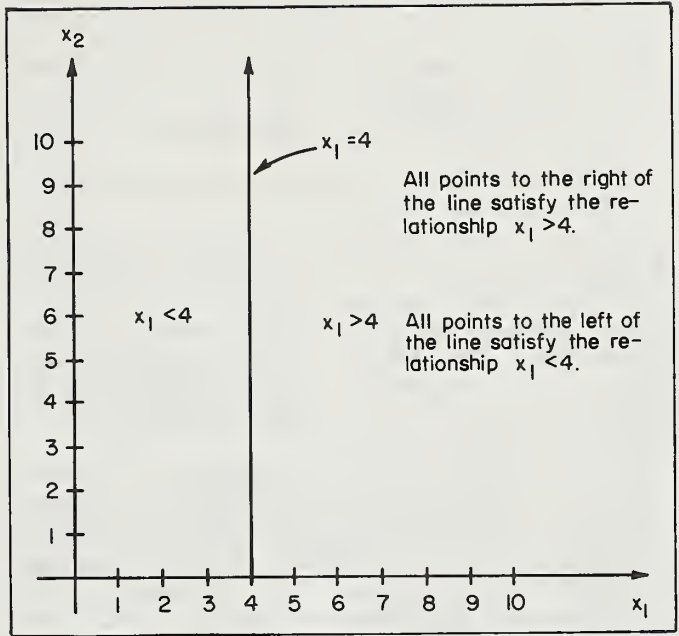


Figure 7.—A plot of the relationship  $x_1 = 4$  showing the location of points where  $x_1 \leq 4$  and  $x_1 \geq 4$ .

in the figure, all points in the first quadrant that satisfy the strict inequality  $x_1 < 4$  lie to the left of the line  $x_1 = 4$ . Also, all points in the first quadrant that satisfy the strict inequality  $x_1 > 4$  lie to the right of the line  $x_1 = 4$ .

We can plot the relationship  $x_2 = 6$  in similar fashion (fig. 8). Note that, in this case, the points in the first quadrant below the line  $x_2 = 6$  satisfy the strict inequality  $x_2 < 6$  and those points in the first quadrant above the line  $x_2 = 6$  satisfy the strict inequality  $x_2 > 6$ . The examples developed thus far can be used to compare the

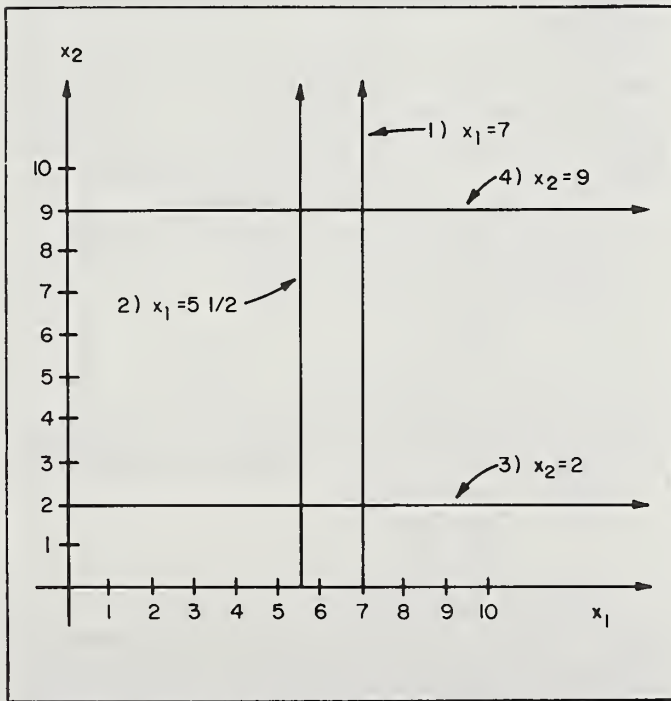


Figure 6.—Plots of the requirements specified in example 1.

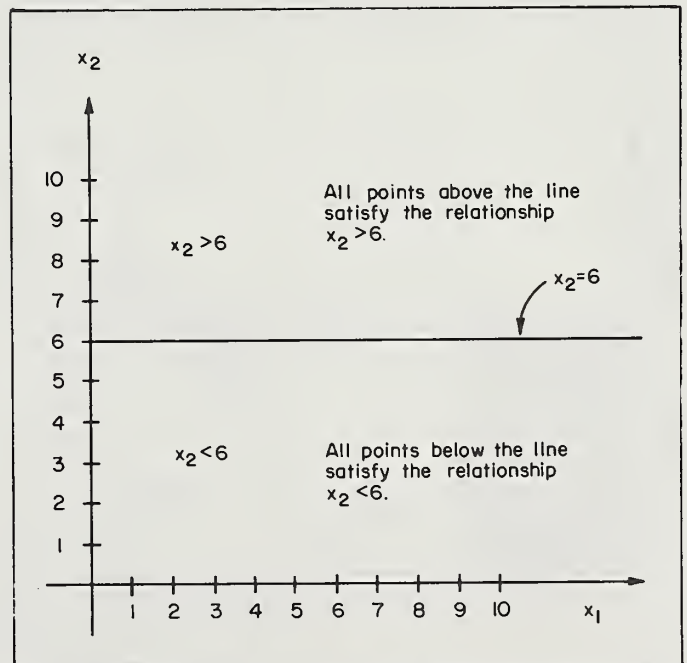


Figure 8.—A plot of the relationship  $x_2 = 6$  showing the location of points where  $x_2 \leq 6$  and  $x_2 \geq 6$ .

plotting of equalities or equations with the plotting of inequalities.

The points that satisfy any inequality can be broken down into two categories:

1. Those that satisfy the equality portion (i.e.,  $x_1 = 4$  or  $x_2 = 6$ );
2. Those that satisfy the strict inequality (i.e.,  $x_1 < 4$ ,  $x_1 > 4$ ,  $x_2 < 6$ ,  $x_2 > 6$ ).

Plotting the points in the first category is equivalent to plotting linear equations in that (assuming the inequality is linear) a line will result.

The points that fall in the second category will consist of all the points below or to the left of the equality line if the strict inequality is  $<$ . On the other hand, if the strict inequality is  $>$ , then the points will be above or to the right of the equality line.

When the points satisfying an inequality are plotted, they consist of all the points satisfying the strict inequality and the points on the line that satisfy the equality.

**Example 2.**—We have indicated that considering only the points in the first quadrant is equivalent to considering only non-negative values for  $x_1$  and  $x_2$ . In inequality notation, the non-negativity restrictions can be expressed as  $x_1 \geq 0$  and  $x_2 \geq 0$ . The points satisfying the equality portions of these constraints are those lying in the positive portions of the  $x_1$  and  $x_2$  axes. The points satisfying the strict inequality portions are those lying in the first quadrant itself.

**Example 3.**—Plot the set of points satisfying the following set of inequalities:

$$x_1 \geq 0 \text{ and } x_1 \leq 4;$$

$$x_2 \geq 0 \text{ and } x_2 \leq 6.$$

This set of points must satisfy all four inequalities simultaneously. It is shown in figure 9 as the shaded area.

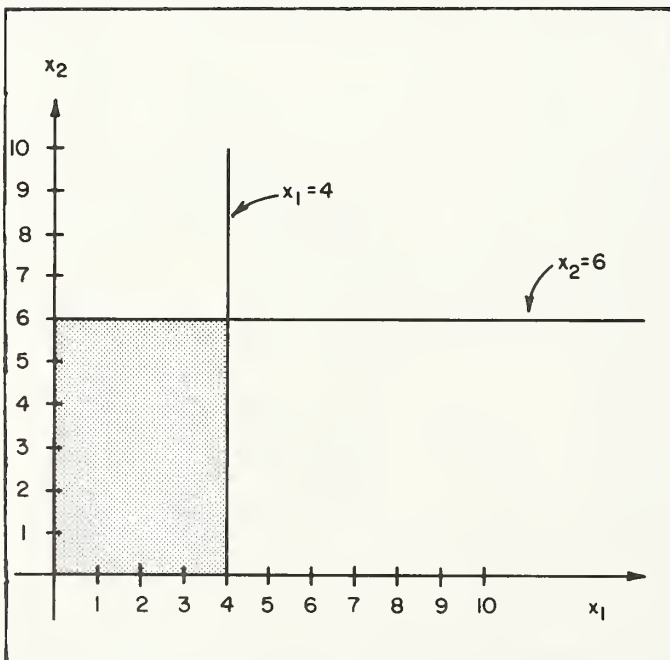


Figure 9.—The set of points satisfying  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $x_1 \leq 4$ , and  $x_2 \leq 6$ .

It is particularly important in understanding linear programming that you understand the way in which sets of points satisfying inequalities are plotted. If you are having problems, study the examples and work the following exercise:

**Exercise 1.**—Plot the sets satisfying the following sets of inequalities:

1.  $x_1 \geq 2$ ,  $x_2 \geq 4$ ,  $x_1 \leq 8$ ,  $x_2 \leq 7$ ;
2.  $x_1 \geq 5$ ,  $x_2 \geq 6$ ,  $x_1 \leq 15$ ,  $x_2 \leq 9$ .

## LINEAR RELATIONSHIPS INVOLVING BOTH $x_1$ AND $x_2$

Until now we have considered relationships involving only  $x_1$  or  $x_2$ . The next two subsections cover linear relationships involving both  $x_1$  and  $x_2$ . This subsection considers equations of the form  $ax_1 + bx_2 = c$ , while the following subsection considers inequalities of the form  $ax_1 + bx_2 \leq c$  or of the form  $ax_1 + bx_2 \geq c$ , where  $a$ ,  $b$ , and  $c$  are specified constants.

### Plotting Equations Involving Both $x_1$ and $x_2$

A relationship between  $x_1$  and  $x_2$  of the form  $ax_1 + bx_2 = c$  is linear; a plot of the set of all points satisfying the relationship will consist of a straight line. You have probably already worked with the algebra of such linear equations. If so, perhaps you recall that once two points on any line are plotted, then the entire line can be plotted. To see how this works, consider the following relationship:

$$3x_1 + 2x_2 = 18$$

There are several methods which we may use to plot the above line.

One of the easiest is to determine the location of the intercepts, or points where the line would intersect the  $x_1$  and  $x_2$  axes if plotted. These points are easy to determine if one recognizes that  $x_2 = 0$  where the line crosses the  $x_1$  axis and  $x_1 = 0$  where the line crosses the  $x_2$  axis.

The intercepts can be located as follows:

1. To find where the line crosses the  $x_1$  axis ( $x_1$  intercept), substitute  $x_2 = 0$  into the equation and solve for  $x_1$ . For our example:

$$3x_1 + 2(0) = 18$$

$$3x_1 = 18$$

$$x_1 = 6$$

and the  $x_1$  intercept is  $x_1 = 6$ ,  $x_2 = 0$ .

2. To find where the line crosses the  $x_2$  axis ( $x_2$  intercept), substitute  $x_1 = 0$  into the equation and solve for  $x_2$ . In our example:

$$3(0) + 2x_2 = 18$$

$$2x_2 = 18$$

$$x_2 = 9$$

and the  $x_2$  intercept is  $x_1 = 0$ ,  $x_2 = 9$ .

Once this calculation is carried out, the line represented by the equation (set of points satisfying the above relationship) can be plotted simply by locating these points and connecting them with a straight line (fig. 10). Note



that, as before, we are only interested in values in the first quadrant.

You may recall seeing the equation for a line expressed in the form  $x_2 = mx_1 + b$  where  $m$  is the slope and  $b$  is the  $x_2$  intercept. Relationships in the form  $ax_1 + bx_2 = c$  can be expressed in the form  $x_2 = mx_1 + b$  by solving for one of the variables in terms of the other. For example, solving for  $x_2$  in terms of  $x_1$  in the equation  $3x_1 + 2x_2 = 18$  yields  $x_2 = (-3/2)x_1 + 9$ . The slope for this example is:

$m = -3/2$  and the  $x_2$  intercept is

$b = 9$  as shown in figure 10.

**Example 4.**—Plot the lines for the equations:

1.  $x_1 + 6x_2 = 9$

2.  $2x_1 + x_2 = 6$

For the first line, the  $x_1$  intercept is:

$$x_1 + 6(0) = 9$$

$$x_1 = 9$$

the  $x_2$  intercept is:

$$1(0) + 6x_2 = 9$$

$$x_2 = 3/2$$

For the second line, the  $x_1$  intercept is:

$$2x_1 + 1(0) = 6$$

$$x_1 = 3$$

the  $x_2$  intercept is:

$$2(0) + 1(x_2) = 6$$

$$x_2 = 6$$

These equations are plotted in figure 11.

### Plotting Inequalities Involving Both $x_1$ and $x_2$

The procedure for plotting points that satisfy inequalities of the forms  $ax_1 + bx_2 \leq c$  and  $ax_1 + bx_2 \geq c$  is basically the same as the one that was used to plot inequalities involving only  $x_1$  or  $x_2$ . That is, we plot the line which represents the equation  $ax_1 + bx_2 = c$ . If the

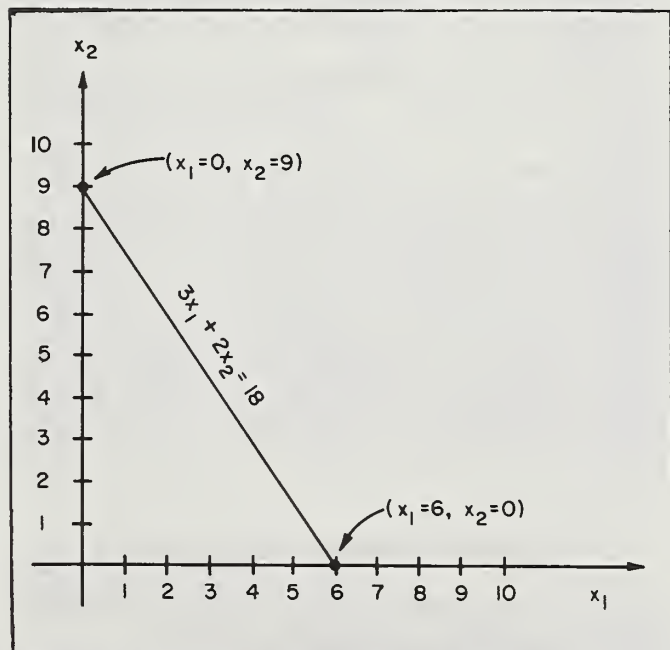


Figure 10.—A plot of the line representing the equation  $3x_1 + 2x_2 = 18$ .

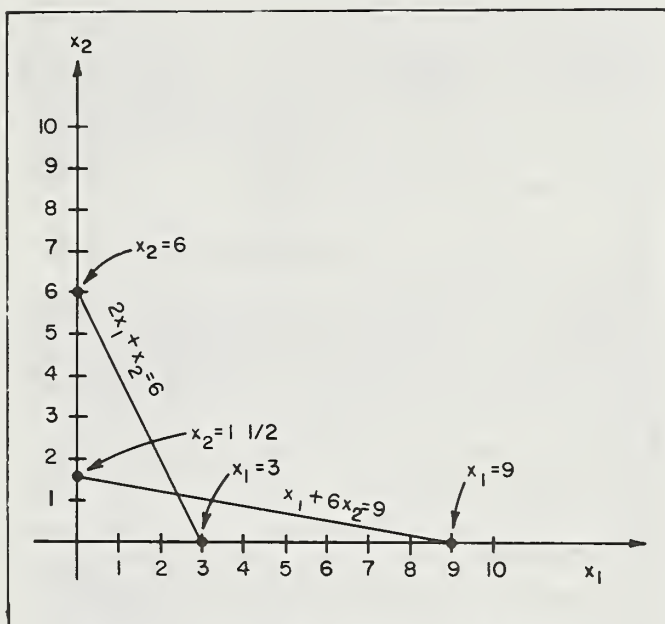


Figure 11.—A plot of the equations from example 4.

inequality is of the  $\leq$  form, all points below the line in the first quadrant also satisfy the inequality. If it is of the  $\geq$  form, then all points above the line in the first quadrant also satisfy the inequality.

To see this, consider the example plotted in figure 10. In this example (shown in fig. 12), all points in area A are going to yield values less than 18 when substituted into the equation.

That is, all points in A satisfy  $3x_1 + 2x_2 < 18$ . Consider the point  $P_1$  denoted in the figure ( $x_1 = 3, x_2 = 2$ ). Substitution of these values into the equation yields:  $3(3) + 2(2) = 13$ .

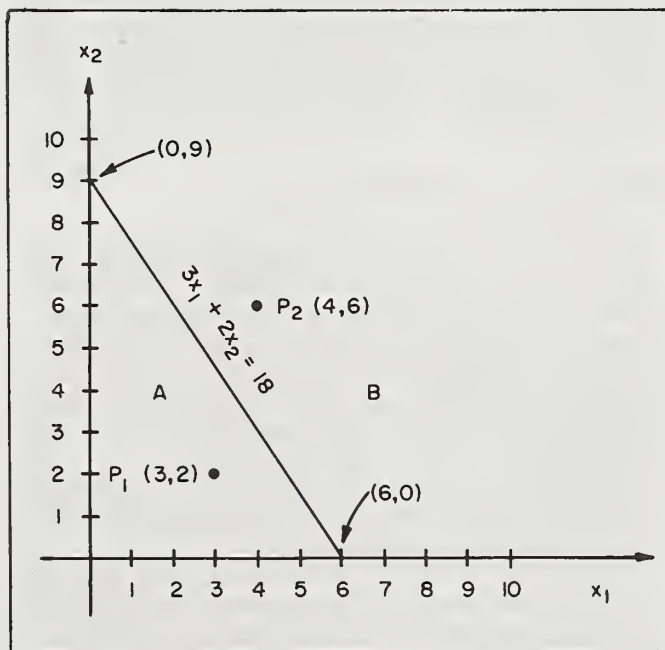


Figure 12.—A plot of a linear equation showing its relation to points that yield smaller and larger values.

Since 13 is less than 18, we have shown that the point  $(x_1 = 3, x_2 = 2)$  satisfies the strict inequality  $3x_1 + 2x_2 < 18$ . In the same manner, it can be demonstrated that all points in area B (above the line) satisfy the inequality  $3x_1 + 2x_2 > 18$ . As an example, consider the point  $P_2$ ,  $(x_1 = 4, x_2 = 6)$  denoted in the figure. Substitution of these values into the equation yields:

$$3(4) + 2(6) = 24$$

Since 24 is greater than 18, the point satisfies the strict inequality  $3x_1 + 2x_2 > 18$ .

Using the above procedure, you should satisfy yourself that all points in area A satisfy the inequality  $3x_1 + 2x_2 < 18$  and that all points in area B satisfy  $3x_1 + 2x_2 > 18$ . Also, all points on the line  $3x_1 + 2x_2 = 18$  satisfy the equation.

**Example 5.**—Plot the set of points that satisfies both of the following inequalities:

$$2x_1 + x_2 \leq 6$$

$$x_1 + 6x_2 \leq 9$$

To do this, note that both inequalities are of the less than or equal to form, as points satisfying the first inequality will be below the line  $2x_1 + x_2 = 6$  and all points satisfying the second inequality will be below the line  $x_1 + 6x_2 = 9$  (see fig. 13). That is, the set of points satisfying the first inequality is located in areas A and B (including boundaries) in figure 13 and the set of points satisfying the second inequality is located in areas B and C (including boundaries) in the same figure. Hence, the set of points that satisfies both inequalities must be in the area denoted by B (including boundaries).

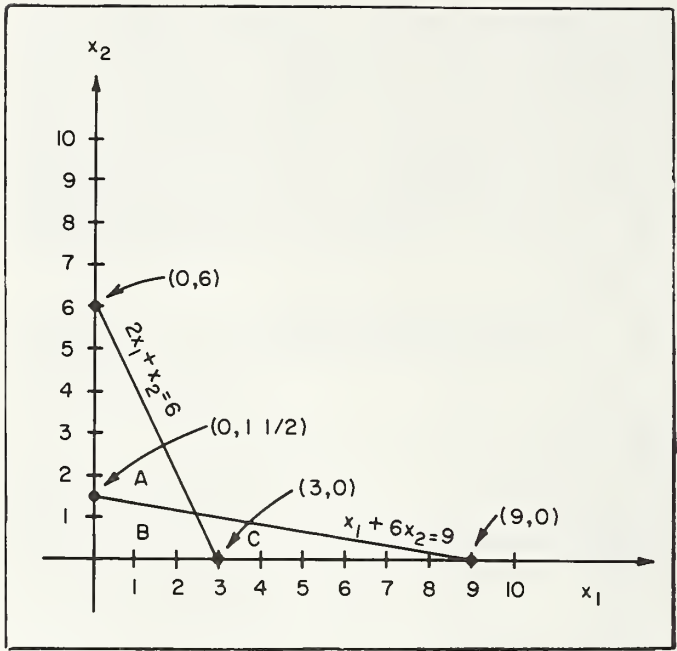


Figure 13.—A plot of the points satisfying the inequalities in example 5.

We will formulate this stocking problem as an LP problem. From the above description, it is clear that the two variables in the problem are the number of cattle and the number of sheep:

$x_1$  = the number of cattle;

$x_2$  = the number of sheep.

The problem is then to determine values for  $x_1$  and  $x_2$  such that the rancher's profit is maximized.

Let us develop an equation which tells us how much profit the ranch will make for any values of  $x_1$  and  $x_2$ . Using the profit information given above, we can write:

$$\text{Profit (\$)} = 60x_1 + 10x_2$$

You should verify this relationship by checking the units of the quantities on the right hand side of the equation.

We have said that we desire to maximize profit which means that we would like to make the quantity  $60x_1 + 10x_2$  as large as possible. The larger  $x_1$  and  $x_2$  are (the more cattle and sheep that are sold), the larger the rancher's profit will be. Such an approach is unrealistic if it ignores any "limits" or "constraints" that the capacity of the ranch places on production of cattle and sheep. In this example, the only limits specified are those arising from the capacity of the ranch for forage production and the requirements of the rancher, his wife, and his hired hands.

### Constraints (Expressed Mathematically)

Now we will consider how we might express these limits or constraints in mathematical form.

**Carrying capacity constraint.**—There are a total of 100 AU's of forage available, which means that the livestock

## INTRODUCTION TO LINEAR PROGRAMMING

This section provides an introduction to linear programming (LP) at an elementary level. The basic concepts and terminology are presented with the aid of an extended example. Comments on notations and additional applications of LP are also mentioned.

### STOCKING A RANCH EXAMPLE

Suppose that a ranch has the equivalent of 100 animal units per year (AU) of forage available for livestock. The rancher has traditionally been a cattleman, but his new wife has a love for sheep. The husband agrees to have at least 100 head of sheep if he can have at least 50 head of cattle. However, his ranch hands threaten to quit if he gets more than 200 sheep on the ranch (Keetch 1977).

**The problem.**—How many cattle and sheep should the rancher stock so that he will maximize profit subject to the restrictions given above?

**Constraints: Additional information.**—One head of cattle consumes one AU of forage per year. Five head of sheep consume one AU of forage per year. Each cow yields a profit of \$60 upon sale. Each sheep yields a profit of \$10 upon sale.



can consume no more than this. From the above information, we see that:

$$\left( \frac{1 \text{ AU}}{\text{head of cattle}} \right) (x_1) + \left( \frac{1/5 \text{ AU}}{\text{head of sheep}} \right) (x_2) =$$

the number of AU's of forage consumed. Since we want to be sure that this does not exceed 100 AU's, we write:

$$1x_1 + (1/5)x_2 \leq 100$$

This inequality expresses the carrying capacity constraint mathematically. (See the Review of Elementary Graphics and Plotting section if you are not familiar with the concept of "inequalities.")

**Wife's sheep requirement.**—Since the wife desires at least 100 sheep, this means that:

$$x_2 \geq 100$$

**Rancher's cattle requirement.**—Since the rancher desires at least 50 cattle, this means that:

$$x_1 \geq 50$$

**Ranch hands' sheep limitation.**—Since the hands will not tolerate more than 200 sheep, this means that:

$$x_2 \leq 200$$

**Logical or non-negativity constraints.**—Since a negative number of cattle or sheep makes no sense, we have the following non-negativity restrictions on  $x_1$  and  $x_2$ :

$$x_1 \geq 0, x_2 \geq 0$$

**Exercise 2.**—In general, for any LP problem, the above logical restrictions hold. In this example, however, they are redundant or not necessary. Why? Now let us write down the complete ranch stocking problem in LP format.

We want to maximize profit ( $60x_1 + 10x_2$ ) subject to the following constraints:

$$\text{Carrying capacity, } x_1 + 1/5x_2 \leq 100;$$

$$\text{Wife's sheep, } x_2 \geq 100;$$

$$\text{Rancher's cattle, } x_1 \geq 50;$$

$$\text{Ranch hands' sheep, } x_2 \leq 200$$

$$\text{Non-negativity, } x_1 \geq 0, x_2 \geq 0.$$

A solution of this problem would consist of the determination of values of  $x_1$  and  $x_2$  (the numbers of cattle and sheep) that will satisfy all of the constraints and that will maximize profit. That is, we want to determine values for  $x_1$  and  $x_2$  yielding a profit equal to or greater than the profit value arising from the selection of any other values for  $x_1$  and  $x_2$  that satisfy all the constraints. This solution is referred to as the optimal solution to the problem because our criterion for optimality is the maximization of the rancher's profit.

## GENERAL REMARKS ABOUT LINEAR PROGRAMMING

Linear programming was first used during World War II; since then, it has been applied to a wide variety of problems. One of the most common definitions of LP is

that it allocates scarce or limited resources among competing activities in an optimal manner. This definition characterizes the nature of problems to which the technique has been applied.

**Exercise 3.**—For the "stocking a ranch" example, identify the:

1. scarce resources;
2. competing activities; and
3. criterion of optimality.

Application areas of LP have included production scheduling, airplane fuel allocation, stock portfolio selection, shipping patterns, and warehouse storage patterns. In forestry, the technique has been applied to problems of timber harvest scheduling, sawmill operation, wood procurement, inventory planning, and land use planning.

**Notation for variables.**—A remark on the notation ( $x_1$  and  $x_2$ ) for the variables denoting numbers of cattle and sheep is in order. You might feel more comfortable with something like:

$c$  = number of cattle

$s$  = number of sheep

because it is easier to associate these variable names with cattle and sheep. If you choose to do this, there is no theoretical problem; the variables may be denoted by any name you desire. The  $x_1, x_2$  notation is used in this report not only because this is the standard LP notation but also because "real world" LP problems often involve hundreds or thousands of variables. A systematic naming system is required in order to keep track of the variables.

## THE GENERAL LINEAR PROGRAMMING MODEL

In this section, we will formulate the general linear programming (LP) model. Components of this model will be discussed and additional LP terminology introduced. When appropriate, the "Stocking a Ranch" example developed in the previous section will be used.

To describe the general model, we will use standard LP notation.

## THE RANCH MODEL

The LP model for the ranch problem is a special case of the general linear programming model in that they both contain all of the same components. Components of the ranch model are reiterated here (fig. 14) to provide analogy and clarification to the discussion about the General Linear Programming Model which follows.

The general model contains  $n$  activity or decision variables and  $m$  operational constraints (not including non-negativity constraints). If we denote the decision variables by:

$$x_1, x_2, \dots, x_n$$

then it is desired to find values for these variables that will maximize

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$



Maximize Profit (Z) = $60x_1 + 10x_2$	
(Objective Function)	
Subject to	$x_1 + 1/5x_2 \leq 100$
	$x_2 \geq 100$
	$x_1 \geq 50$
The following	
Operational	
Constraints	$x_2 \leq 200$
	$x_1 \geq 0, x_2 \geq 0$

Figure 14.—Components of the “stocking a ranch” model.

This set of values must satisfy both the  $m$  operational constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1;$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2;$$

.

.

.

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m;$$

and the  $n$  non-negativity constraints:

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

The notation used for the general model is as follows.

The  $c_j$ 's ( $j = 1, 2, \dots, n$ ) are the coefficients of the decision variables ( $x_j$ ) in the objective function (e.g.,  $3x_2$ ). The index  $j$  is the counting index or index of summation for the decision variables ( $j = 1$  means  $x_1$ , etc.).

The  $b_i$ 's ( $i = 1, 2, \dots, m$ ) are the right hand side values (RHS) of the constraints. The index  $i$  is the counting index for the constraints ( $i = 1$  means the first constraint, etc.).

Finally, the  $a_{ij}$ 's are the coefficients of the decision variables on the left hand side (LHS) of the constraints ( $a_{ij}$  is the coefficient of  $x_j$  in constraint  $i$ ;  $i = 1, 2, \dots, m$ ;  $j = 1, 2, \dots, n$ ).

It should be noted that an important, often violated assumption of LP is that all of the coefficients in the model are known constants. This assumption will be discussed in more detail in the Advantages and Disadvantages of Linear Programming section.

A remark about the type of inequalities or operational constraints in the general model is in order. You will note that in the general model, all of the operational constraint inequalities are of the less than or equal to ( $\leq$ ) form while the ranch problem model has two operational constraints of the greater than or equal to ( $\geq$ ) form. The general model is given in what is known as *standard* or *canonical* form, whereas the ranch model is not in this form because of the  $\geq$  constraints. Such deviations from standard form are common and, as will be seen later, they do not have a significant effect on problem solution.

Other frequently encountered situations are objective functions that are minimized and constraints in the form of equalities ( $=$ ).

Now let us discuss each of the components of the LP model.

## OBJECTIVE FUNCTION—QUANTIFIES THE CRITERION OF OPTIMALITY

$$(Z) = 60x_1 + 10x_2; \text{ or}$$

$$(Z) = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

This function is sometimes called the criterion function, and the associated variable  $Z$  is the criterion variable. This function serves to quantify the criterion of optimality for a given problem. It does this by assigning a numerical value (in the ranch problem, a profit value) to each solution or possible set of values on the decision variables.

In the ranch problem, the criterion of optimality is the maximization of profit, and the objective function determines the profit resulting from any values of  $x_1$  and  $x_2$ .

In the general model, the coefficients  $c_1, c_2, \dots, c_n$  measure the contribution per unit of each decision variable to the optimality criterion.

For example, in the ranch problem,  $c_1 = \$60$  and  $c_2 = \$10$  represent the profit per head of cattle and sheep, respectively.

We have expressed the objective function in terms of all of the decision variables in the general model, but in many problems, some of these variables may not appear in this function (see the discussion of goal programming in the Advantages and Disadvantages of Linear Programming section). This is analogous to stating that some of the  $c_j$ 's may be zero.

While profit maximization is one of the most frequently used criteria of optimality, it is not the only one encountered. For example, objective functions for land use planning problems are often stated in terms of cost minimization, maximization of present net return, or maximization of the production of some product such as water, forage, or timber.

## CONSTRAINTS

The constraints in the LP model have been broken into two types, operational constraints and non-negativity constraints.

Operational constraints arise from the nature of the system the model represents. For example, in the ranch problem, the first operational constraint reflects the capability of the system to produce forage. It is included in the problem because the ranch has a limit on the amount of forage it can produce, and we want to limit the production of cattle and sheep accordingly. The remaining operational constraints in the ranch problem reflect the desires of the people associated with the ranch. Further examples of the variety of forms operational constraints can take will be described later.

The non-negativity constraints are logical constraints reflecting the impossibility of having negative quantities of the commodities that the decision variables represent. For example, in the ranch problem, these constraints reflect the impossibility of having negative numbers of cattle and sheep. If you worked exercise 3, you realized that the second and third operational constraints in the ranch problem make the non-negativity constraints

redundant or unnecessary. This is not unusual, for in many problems, operational constraints make the non-negativity constraints for some of the variables redundant. Most computer LP packages automatically assume that they are part of the problem. As a result, the user does not need to specify them as data to be input into the computer.

In the general model, the  $a_{ij}$ 's ( $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ ) were used to denote the coefficients of the decision variables on the left hand side (LHS) of the constraints. As we have seen in the ranch example,  $i = 1, 2, 3, 4$ , (4 operational constraints) and  $j = 1, 2$  (2 decision variables).

Referring to figure 14, we see that for the first operational constraint ( $i = 1$ ), we have  $a_{11} = 1$  and  $a_{12} = 1/5$ . Similarly, for the second operational constraint ( $i = 2$ ),  $a_{21} = 0$  and  $a_{22} = 1$ . Note that some of the  $a_{ij}$ 's can be zero. In most land management planning problems, the majority of the  $a_{ij}$ 's are zeros.

In the general model, the  $b_i$ 's ( $i = 1, 2, \dots, m$ ) were used to denote the RHS values for the operational constraints. In the ranch example,  $b_1 = 100$ ,  $b_2 = 100$ ,  $b_3 = 50$ , and  $b_4 = 200$ . Each operational constraint corresponds to a limitation on some resource and the right hand side (RHS) values represent the limiting amount of these resources. In the ranch problem, forage is a limited resource; there are at most 100 AU's (the RHS of the forage constraint) available.

## DECISION (ACTIVITY) VARIABLES

In any LP problem, the constraints and the objective function are linear functions of a set of variables referred to as decision or activity variables. Specifying values for decision variables is equivalent to specifying a management strategy in terms of the levels and types of activities that will be implemented. In the ranch problem, the decision variables represent the number of cows ( $x_1$ ) and the number of sheep ( $x_2$ ) to be stocked on the ranch. An example of a management strategy would be to stock 50 head of cattle ( $x_1 = 50$ ) and 100 head of sheep ( $x_2 = 100$ ). In general, for any LP problem, a strategy is specified when a non-negative value (some of these values may be zero) is assigned to every decision variable in the problem.

Linear programming derives its name from the fact that all mathematical relationships, as expressed by the objective function and constraints, are linear. That is, they contain no terms involving either powers of the  $x$ 's other than one, or cross products of the  $x$ 's. For example, the expressions:

$x_1 + (1/5) x_2^2$  and  $x_1^2 + (1/5) x_1 x_2$  are nonlinear in  $x_1$  and  $x_2$ . On the other hand, all relationships in the general LP model and in the ranch problem are linear in the decision variables.

## DEFINITIONS

We defined a *strategy* for any LP problem as a set of values assigned to the decision variables associated

with the problem. The constraints associated with the problem define what is called the set of *feasible strategies*.

A *feasible strategy* is one that satisfies all of the constraints in the problem. The set of feasible strategies is simply the set of all that satisfy all the restraints.

Since the objective function quantifies the optimality criterion for an LP problem, it determines which of the feasible strategies is the optimal strategy.

The *optimal strategy* for an LP problem is the feasible strategy that maximizes (or minimizes) the objective function. When the values of the  $x$ 's corresponding to the optimal strategy are substituted into the objective function, the resultant value of  $Z$  is at least as large as (or at least as small as) the value of  $Z$  which results when the  $x$  values corresponding to any other feasible strategy are substituted.

A linear programming problem is solved by determining which strategy in the set of feasible strategies is optimal. That is, the strategy that either maximizes or minimizes the objective function is determined. A general discussion of how this is done is presented in the Land Use Problem and Some Elementary Sensitivity Analysis section of this report.

**Example 6.**—With reference to the ranch example, are the following strategies feasible or non-feasible?

1.  $x_1 = 60$  cattle,  $x_2 = 150$  sheep
2.  $x_1 = 75$  cattle,  $x_2 = 90$  sheep
3.  $x_1 = 45$  cattle,  $x_2 = 175$  sheep

1. This is a feasible solution since 60 cattle exceed the rancher's minimum of 50, and the 150 sheep lie between the wife's lower limit of 100 sheep and the ranch hands' upper limit of 200 sheep. Also,  $1(60) + 1/5(150) = 60 + 30 = 90$ , which is less than the upper limit of forage of 100 AU's.

2. This solution is infeasible because 90 sheep is less than the wife's minimum requirement of 100 sheep.

3. This solution is infeasible because 45 cattle is less than the rancher's minimum requirement of 50 cattle.

Note that once a strategy violates one constraint, it is not necessary to examine it further to determine infeasibility because a feasible strategy must satisfy all the constraints.

## MORE LINEAR PROGRAMMING TERMINOLOGY—MATRIX AND VECTOR

The notation used above for the general LP model is standard for most disciplines that utilize the technique, and as a result, some additional terminology has been developed. To introduce this terminology, we need the following definitions:

A *matrix* is a rectangular array of numbers having a certain number of rows ( $m$ ) and columns ( $n$ ) where  $m$  and  $n$  are arbitrary integers.

A *vector* is a matrix composed of only one row (row vector) or one column (column vector) of numbers or variables.



**Example 7.**

1.  $\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & 0 & 3 & 9 \\ 4 & 16 & 0 & 7 \end{bmatrix}$  This is an  $m = 3$  row and  $n = 4$  column matrix ( $3 \times 4$ )
2.  $[1 \ 6 \ 1 \ 2 \ 9]$  This is a 5-element row vector ( $1 \times 5$ )
3.  $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \\ 0 \\ 0 \\ 6 \end{bmatrix}$  This is a 7-element column vector ( $7 \times 1$ )

In all three examples, the numbers in parentheses represent the number of rows and columns and are called the dimensions of the matrix or vector.

The terms "matrix" and "vector" are introduced here because they are frequently encountered in discussions of LP. In particular, consider the following four cases.

**Case 1.—The Vector of Objective Function Coefficients**

This vector (in row form) is  $[c_1, c_2, \dots, c_n]$  for the general LP model and is a  $1 \times n$  vector (one coefficient in the objective function for each of the  $n$  activity variables). This vector is sometimes called the cost vector or price vector.

**Example 8.**—In the ranch problem, the price vector is  $[60, 10]$  and is a  $1 \times 2$  vector.

**Case 2.—The Vector of Decision Variables**

This vector (in row form) is  $[x_1, x_2, \dots, x_n]$  for the general LP model and is a  $1 \times n$  vector.

**Example 9.**—In the ranch problem, the vector of decision variables is  $[x_1, x_2]$  and is a  $1 \times 2$  vector.

**Case 3.—The Matrix of Coefficients of Activity Variables on the Left Hand Sides of the Constraints**

In the general LP model, this will be an  $m$  row ( $m$  constraints) by  $n$  column ( $n$  activity variables) matrix. That is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The subscripts on the elements ( $a$ 's) in the matrix identify the row and column address for each element. This matrix is called the  $A$  matrix, the activity matrix, or the technological matrix.

**Example 10.**—In the ranch problem (fig. 14), the  $A$  matrix is:

$$A = \begin{bmatrix} 1 & 1/5 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that this matrix has four rows (one for each of the four operational constraints) and two columns (one for each of the two activity variables).

**Case 4.—The Vector of Right Hand Sides for the Constraints**

In the general LP model, this vector (in column form) will consist of  $m$  elements (one for each operational constraint). That is:

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This vector is called the resource availability vector, the requirement vector, the RHS vector, or most often, the  $B$  vector.

**Example 11.**—In the ranch problem, the  $B$  vector is:

$$B = \begin{bmatrix} 100 \\ 100 \\ 50 \\ 200 \end{bmatrix}$$

Note that this vector has four elements, one for each operational constraint in the problem.

If you have had any exposure to LP, you have probably heard people talking about "the matrix" or "the  $A$  matrix," and "the  $B$  vector" or "the RHS vector." This terminology arises from the LP notation just described.

**THE MATRIX APPROACH TO EXPRESSING THE LP MODEL**

Up to now, we have expressed linear programming models in their equation or inequality form. There is another way of writing a linear programming problem which is called the matrix or tableau approach.

To see how this is done, consider the ranch problem. This problem is displayed in matrix form in figure 15. Note that in this form, the constraints and decision variables are identified as rows and columns, respectively. The tableau or matrix contains only the coefficients of the decision variables, the constraint types, and the right hand side values. The main difference between the two formats is that, in the matrix approach, one does not write down the decision variables associated with each of the coefficients. For a small problem like the ranch



Rows (constraint types)	Columns (decision variables)	Number of Cattle ( $x_1$ )	Number of Sheep ( $x_2$ )	Constraint Type	RHS
Forage Availability		1	1/5	$\leq$	100
Wife's Sheep Requirement			1	$\geq$	100
Rancher's Cattle Requirement		1		$\geq$	50
Ranch Hand's Sheep Requirement			1	$\leq$	200
Objective Function		60	10	-	-

Figure 15.—A matrix or tableau representation of the ranch problem.

problem, it takes no more time to write down the problem in equation form than in matrix form. However, for large problems involving hundreds or thousands of matrix entries, the extra time required to write down the appropriate activity variable with each coefficient is considerable. Thus, the matrix form offers time-saving advantages.

In figure 15, note that where a decision variable did not appear in a particular constraint row, instead of entering a zero, the cell was left blank. The A matrix for a typical land management problem will have a great many such empty cells because many constraint rows involve only a small number of the decision variables defined for the problem (i.e., as mentioned before, many of the  $a_{ij}$ 's = 0). Leaving the large number of cells that contain zero coefficients blank saves time and enhances the ease with which the matrix can be interpreted.

One result of the use of the matrix format has been the increasing use of the term "row" to mean constraint, and the term "column" to mean decision variable. It is important to realize that when you hear a statement like "the coefficient for sheep in the forage row is 1/5," it means that the coefficient of the decision variable representing number of sheep (in this example  $x_2$ ) in the forage constraint (in this example, row 1 in the matrix) is 1/5 (fig. 15).

It is also important to realize that the numbers in a linear programming matrix really represent the coefficients of the decision variables in inequality or equality constraints. Thus, when you change these coefficients or add new ones, you are either making changes in existing constraints or are adding new constraints to the problem.

**Example 12.**—Consider the ranch problem again. Suppose that the rancher has received the latest extension research bulletin which contains an article that suggests that sheep consume 1/4 of a cattle AU of forage per year. Also, the rancher has determined that if he cannot make at least \$2,000 per year, he would be better off selling his ranch. How do we reformulate the ranch example to account for this new information?

Rows (constraint types)	Columns (decision variables)	Number of Cattle ( $x_1$ )	Number of Sheep ( $x_2$ )	Constraint Type	RHS
Forage Availability		1	1/4	$\leq$	100
Wife's Sheep Requirement			1	$\geq$	100
Rancher's Cattle Requirement		1		$\geq$	50
Ranch Hand's Sheep Requirement			1	$\leq$	200
Rancher's Profit Requirement		60	10	$\geq$	2000
Objective Function		60	10	-	-

Figure 16.—A matrix or tableau representation of the modified ranch problem.

1. For the change in sheep forage consumption rate, all we need to do is change the entry (1/5) in the forage row for sheep to 1/4. The new forage constraint is  $x_1 + (1/4)x_2 \leq 100$ .
2. The profit restriction must be incorporated into the model through the addition of a new constraint which will be the equivalent of adding a new row to the matrix. This new constraint, expressed in inequality form, is:

$$60x_1 + 10x_2 \geq 2,000.$$

The matrix representation of this modified ranch problem is presented in figure 16. The changed coefficient and the added constraint have been circled.

## THE PROBLEM OF UNITS

Until this point, we have talked about the constraint equations and inequalities without concern for the units that are associated with them. You have undoubtedly considered the problem of units for equations. Suppose we have an equation of the form  $A = B$ . For the equality to make any sense (i.e., to avoid the problem of equating apples and oranges), the quantity A must be in the same units as quantity B.

This idea carries over to inequalities in the same way. That is, an inequality like  $A < B$  or  $A > B$  makes no sense unless the quantities A and B are in the same units. To explore this further, let us examine the units of some of the constraints in the ranch problem. Consider the forage constraint:

$$x_1 + (1/5)x_2 \leq 100$$

The units that are implied in this relationship are:

$$\left( \frac{1 \text{ AU}}{\text{head of cattle}} \right) \left( x_1 \text{ head of cattle} \right) + \left( \frac{1/5 \text{ AU}}{\text{head of sheep}} \right) \left( x_2 \text{ head of sheep} \right) \leq 100 \text{ AU}$$

Note that the units “head of cattle” and “head of sheep” cancel out so that the quantity of  $x_1 + (1/5)x_2$  is expressed in terms of AU’s. Thus, we have the same units on each side of the inequality sign.

As another example, consider the constraint representing the rancher’s cattle requirement:

$$x_1 \geq 50$$

In this case, the units of  $x_1$  are cattle and the RHS of 50 also represents cattle so that the two quantities are already in the same units. The units that are implied in this relationship are:

$$\left( \frac{1 \text{ head/cattle}}{\text{head/cattle}} \right) \left( x_1 \text{ head/cattle} \right) \geq 50 \text{ head/cattle}$$

Again, the units are the same on each side of the inequality sign.

**Exercise 4.**—Consider the modified ranch problem represented above. Analyze the constraints representing:

1. the wife’s sheep requirement;
  2. the ranch hands’ sheep requirement; and
  3. the rancher’s profit requirement;
- and verify that the quantities on each side of the inequality signs have the same units.

As a final note on units, recall the matrix representation of an LP problem described previously. As we pointed out earlier, each row in the matrix corresponds to a constraint, and each column corresponds to an activity variable. From the above discussion on units, we conclude:

1. Each activity variable (column) has an associated unit;
2. Each constraint coefficient has an associated unit; and
3. Each constraint (row) is expressed in terms of some unit that depends on the units in 1 and 2.

In developing the matrix for any linear programming problem, it is important to always keep the units in mind. We will talk more about this later as we investigate other examples.

## A GRAPHICAL INTERPRETATION OF THE “STOCKING A RANCH” PROBLEM

We have been talking about the “stocking a ranch” example long enough now that it probably seems as though we are flogging a dead horse. There remains, however, one important aspect that we have not covered, and that is a graphical interpretation of the problem. Before we begin, a short discussion of the importance of this material is in order.

For small linear programming (LP) problems, like the ranch problem, involving two decision variables, we can use a simple coordinate system to graphically describe exactly what is happening in the problem. This effort serves at least two useful purposes:

1. We can plot the constraint inequalities and, hence, plot the set of feasible solutions.

2. We can gain some insight into the procedure by which an LP problem is solved.

The graphical approach is generally limited in application to two-variable problems. It is difficult (but possible) to plot in three dimensions (three decision variables), and it is impossible to plot in four or more dimensions (four or more decision variables). Despite this limitation, graphical interpretation is still a useful technique because the insights into LP that it provides for two-variable problems are equally applicable to larger problems. That is, the mathematical relationships encountered in a simple two-variable problem generalize to problems with any number of decision variables. Consider the ranch example. (If you are not comfortable with the concept of plotting equalities and inequalities, you should review the material presented in the Review of Elementary Graphics and Plotting section of this report.)

We have already indicated that any solution to or strategy for the ranch problem consists of the assignment of numerical values to the decision variables ( $x_1$  = number of cattle and  $x_2$  = number of sheep). Thus, we can use an  $x_1, x_2$  coordinate system (see fig. 2) to represent all strategies or combinations of values for  $x_1$  and  $x_2$ .

As mentioned in previous sections, in most situations, the decision variables for an LP program can take on only positive values. Certainly this is true for the ranch example, since negative numbers of cattle and sheep have no meaning. Mathematically, these logical restrictions are represented by the non-negativity constraints described in the Introduction to Linear Programming section. That is,  $x_1 \geq 0$  and  $x_2 \geq 0$ .

From a plotting standpoint, these non-negativity restrictions enable us to ignore all quadrants of the coordinate system in figure 2 except the first. That is, the non-negativity constraints will eliminate strategies that do not correspond to points in the first quadrant.

For purposes of reference, the complete, unmodified ranch problem model is as follows:

$$\begin{aligned} \text{Max profit (Z)} &= 60x_1 + 10x_2 \\ \text{Subject to: } x_1 + (1/5)x_2 &\leq 100 \\ x_2 &\geq 100 \\ x_1 &\geq 50 \\ x_2 &\leq 200 \\ x_1 \geq 0, x_2 &\geq 0. \end{aligned}$$

As with the non-negativity constraints, the operational constraints have a graphical interpretation. Recall that in the Introduction to Linear Programming section, we indicated the following:

1. The operational constraints define a set of feasible strategies where a feasible strategy is defined to be one that satisfies all of the constraints simultaneously.
2. The optimal strategy is the strategy in the set of feasible strategies that maximizes profit (in this example) as expressed by the objective function.

We are going to divide the discussion of the graphical interpretation of the ranch problem into two parts corresponding to the two concepts listed above.



## DETERMINING THE SET OF FEASIBLE SOLUTIONS

Our procedure consists of plotting each operational constraint in the ranch problem. Once all of the constraints are plotted, we can delineate the set of feasible solutions for the problem.

### Wife's Sheep Constraint

To begin, consider the wife's sheep requirement constraint,  $x_2 \geq 100$ . The set of points (strategies) that satisfy this constraint is plotted in figure 17. This set is represented by the shaded area in the figure, which consists of all points in the first quadrant with an  $x_2$  coordinate of 100 or more. All of the points below the line  $x_2 = 100$  are eliminated from consideration as feasible solutions because they violate the wife's sheep requirement. All of the points above  $x_2 = 100$  are feasible solutions as far as the wife is concerned.

### Ranch Hands' Constraint

Now consider the impact of the ranch hands' sheep restriction,  $x_2 \leq 200$ . Figure 18 shows a plot of the set of points that satisfies both the ranch hands' and the wife's sheep requirements. This requirement eliminates any strategy having more than 200 sheep ( $x_2 > 200$ ); hence, all the points above the  $x_2 = 200$  violate this constraint and, therefore, cannot be feasible solutions. The two constraints in combination restrict possible feasible strategies to those that are represented by points in the shaded area in figure 18.

**Exercise 5.**—Are the points lying on the lines  $x_2 = 100$  and  $x_2 = 200$  feasible strategies at this point? Why or why not?

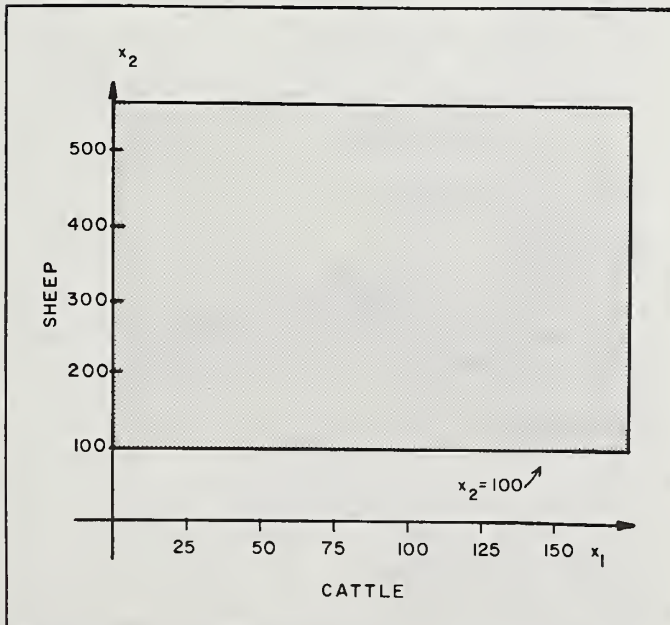


Figure 17.—The set of strategies satisfying the wife's sheep requirement.

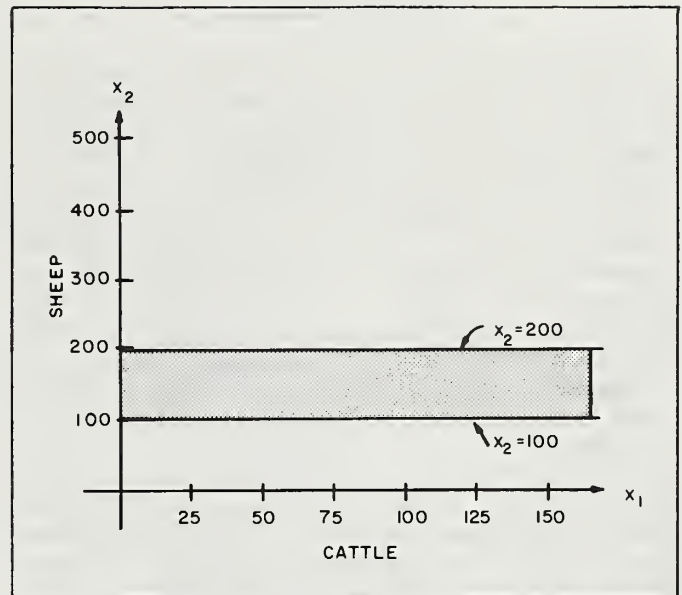


Figure 18.—The set of strategies satisfying both the wife's and the ranch hands' sheep requirement.

### Ranchers' Constraint

The next constraint that we will consider represents the rancher's cattle requirement,  $x_1 \geq 50$ . The set of points that satisfies this requirement, and the wife's and ranch hands' requirements is represented by the shaded area in figure 19. Only points to the right of the line (as well as the line itself) represent strategies that satisfy the rancher's requirement of at least 50 cattle. The addition of the rancher's cattle requirement has had the effect of eliminating all strategies in the rectangle labeled "A" in figure 19.

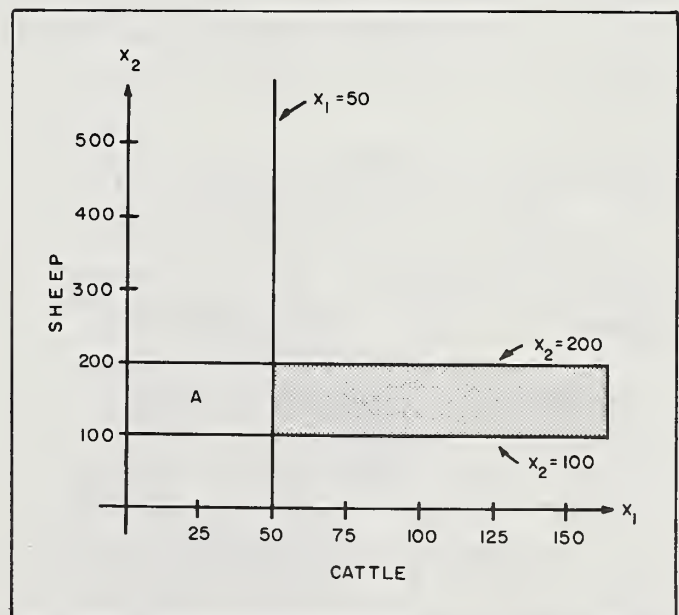


Figure 19.—The set of strategies satisfying the wife's, the rancher's, and the ranch hands' stock requirements.



## Forage Constraint

Finally, let us consider the forage limitation constraint,  $x_1 + (1/5)x_2 \leq 100$ . This constraint is a function of both  $x_1$  and  $x_2$ , so the plotting procedure (see the Review of Elementary Graphics and Plotting section) is different from that used for the previous constraints. Since it is a  $\leq$  constraint, only strategies represented by points on or below the line  $x_1 + (1/5)x_2 = 100$  will satisfy it. Hence, the only strategies that satisfy this constraint and the other three constraints are those represented by points in the shaded trapezoid in figure 20. This is the set of feasible solutions for the ranch problem.

**Exercise 6.**—Select two or three points in the shaded area in figure 20 and verify that they satisfy all of the constraints.

Careful study of the above plotting process can provide insight into the role of the constraints in any LP problem. Each has the effect of eliminating some strategies from consideration as feasible strategies because they violate that constraint. When all constraints are considered together, there may be a set of strategies that have not been eliminated. This set, if it exists, consists of strategies that are feasible for all of the constraints.

It is possible to have a set of constraints such that there are no strategies that satisfy all of them, and in such cases, there is no feasible strategy. This problem will be discussed in more detail in the Land Use Problem and Some Elementary Sensitivity Analysis section of this report. While we cannot represent the situation graphically, the same relationships between constraints and feasible and infeasible strategies hold for LP problems with any number of decision variables.

**Exercise 7.**—Verify graphically that there are no strategies (points) that satisfy all of the following constraints:

$$x_1 \geq 0$$

$$x_2 \geq 0$$

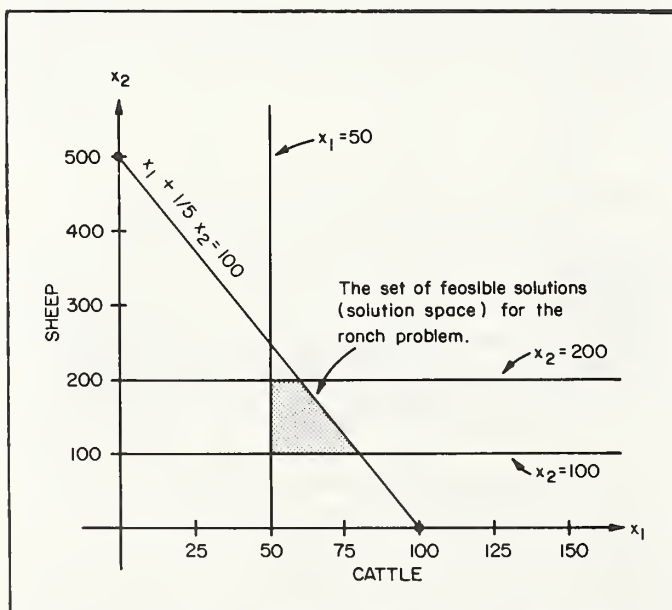


Figure 20.—The set of feasible solutions for the ranch problem.

$$x_2 \leq 100$$

$$x_2 \geq 200$$

Note the similarity between these constraints and some of those in the ranch problem.

## DETERMINATION OF THE OPTIMAL SOLUTION

In this subsection, we will describe a graphical process for identification of the optimal solution for the ranch problem. This process is limited in application to problems where the set of feasible solutions can be represented graphically, i.e., for problems having only two decision variables. However, the process is useful because it shows how the nature of the objective function determines which member of the set of feasible solutions is the optimal solution. It is also useful for elaboration of some of the underlying principles of the general methodology used to solve linear programming problems.

### Determining the Objective Function

We will begin the description of this process by examining the objective function  $Z = 60x_1 + 10x_2$ . Recall that  $Z$  is the criterion variable and, in this case,  $Z$  = profit.

We want to select the strategy contained in the set of feasible strategies in figure 20 that will maximize  $Z$ . Note that since our objective function is linear, by plotting it for different trial values of  $Z$ , we will obtain a series of parallel straight lines.

To see this, consider the following values for  $Z$  and the corresponding lines that result when the objective function is plotted.

**Example 13:**

$$Z = 1,000 = 60x_1 + 10x_2$$

$$\text{Solve for } x_2: x_2 = 100 - 6x_1$$

For this line, the  $x_1$  intercept is:  $x_2 = 0$ ,

$$x_1 = \frac{100}{6} = 16 \frac{2}{3}$$

The  $x_2$  intercept is:

$$x_1 = 0, x_2 = 100.$$

**Example 14:**

$$Z = 3,000 = 60x_1 + 10x_2$$

$$\text{Solve for } x_2: x_2 = 300 - 6x_1$$

For this line, the  $x_1$  intercept is:  $x_2 = 0$ ,  $x_1 = 50$

The  $x_2$  intercept is:

$$x_1 = 0, x_2 = 300.$$

**Example 15:**

$$Z = 6,000 = 60x_1 + 10x_2$$

$$\text{Solve for } x_2: x_2 = 600 - 6x_1$$

For this line, the  $x_1$  intercept is:

$$x_2 = 0, x_1 = 100$$

The  $x_2$  intercept is:

$$x_1 = 0, x_2 = 600.$$

This procedure can be carried out for any value of  $Z$ , but the three we examined will suffice for our purposes. Using the techniques described in the Review of Element-

tary Graphics and Plotting section, the three lines are plotted in figure 21.

Note that these lines all have the same slope; all lines resulting from any choice of value for  $Z$  will have this slope. We know that for each of these lines all points on the line yield the same value for  $Z$ . For example, all points on the line  $Z = 6000 = 60x_1 + 10x_2$  have coordinate values that, when substituted into the objective function, yield a profit of \$6000. In terms of strategies, this means that all strategies that yield a profit of \$6000 correspond to points on the line  $6000 = 60x_1 + 10x_2$ . The same remarks apply for the lines corresponding to  $Z = 1000$ , 3000, or any other value.

Now we can relate these lines to the set of feasible solutions developed earlier in figure 20. None of the three lines resulting from our selection of trial values for profit intersect this set (fig. 22). Since each of these lines corresponds to the strategies that yield profits of \$1,000, \$3,000, and \$6,000, respectively, the fact that these lines fail to intersect the set of feasible solutions suggests that all of these trial profit levels are infeasible.

**Exercise 8.**—Select one of the trial objective function lines plotted in figure 22 and satisfy yourself that none of the points on that line correspond to feasible solutions.

We obtained our three plots of the objective function by selecting trial profit levels or values of  $Z$ . The above discussion implies that if a profit trial value is feasible for the ranch problem, then the line that results when the objective function is plotted should intersect the set of feasible solutions (i.e., some of the points on this line correspond to feasible strategies for the problem).

Consider the lines corresponding to  $Z = 1,000$  and  $Z = 3,000$ . These lines represent strategies that yield profits less than the smallest feasible profit.

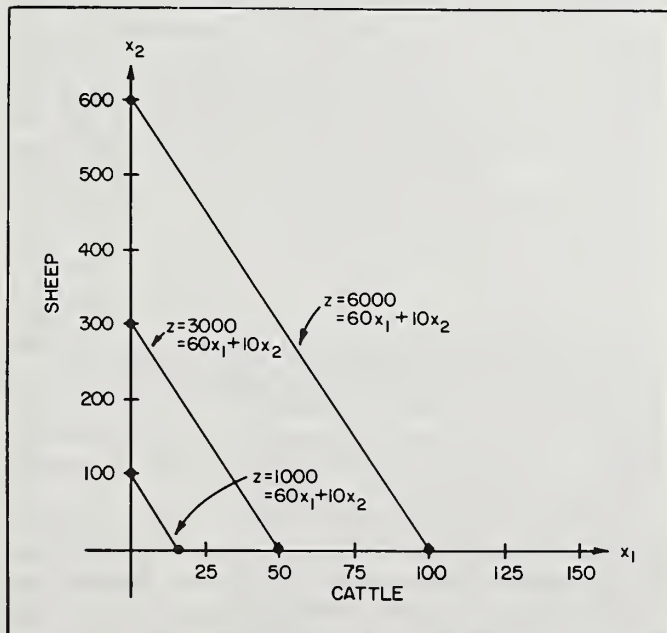


Figure 21.—Plot of the objective functions of the ranch problem for three values of profit ( $Z$ ).

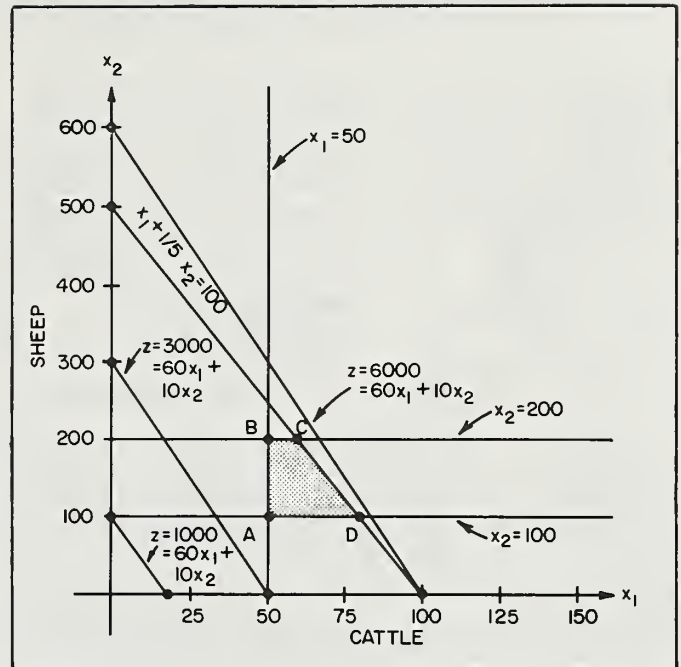


Figure 22.—Plot of the set of feasible solutions and some trial objective function values.

Consider the strategy represented by point A in figure 22. This point corresponds to the intersection of the  $x_1 = 50$  and  $x_2 = 100$  lines. The profit for this strategy is:

$$Z = 60(50) + 10(100) = \$4,000$$

This is a feasible strategy, and by examination of the figure, it is easy to see that any other feasible strategy will yield a greater profit.

By following the same reasoning, we can see that strategies represented by the line  $Z = \$6,000$  yield a profit that is greater than feasible. These facts suggest that if we start with small trial values of  $Z$  and steadily increase them, the following will happen:

1. Initial profit levels defined by the trial values of  $Z$  will be less than the minimum feasible amount (4,000).
2. As trial values increase beyond 4,000, profit levels will be feasible and the lines will intersect the set of feasible solutions (fig. 23).
3. At some trial value (in this case  $Z = \$5,800$ ), the corresponding line will intersect the set of feasible solutions at only one point (in this case the point D). This point will correspond to the optimal strategy that maximizes profits (fig. 23).

In general this is true, but it is possible that the line will intersect one of the boundary segments of the feasible set. Then there are an infinite number of optimal solutions. This case will be discussed in the Land Use Problem and Some Elementary Sensitivity Analysis section.

4. As larger trial values of  $Z$  (for example,  $Z = 6,000$ , fig. 22) are chosen, the line moves out beyond the set of feasible solutions. Profit levels corresponding to these lines are infeasible and greater than optimal.



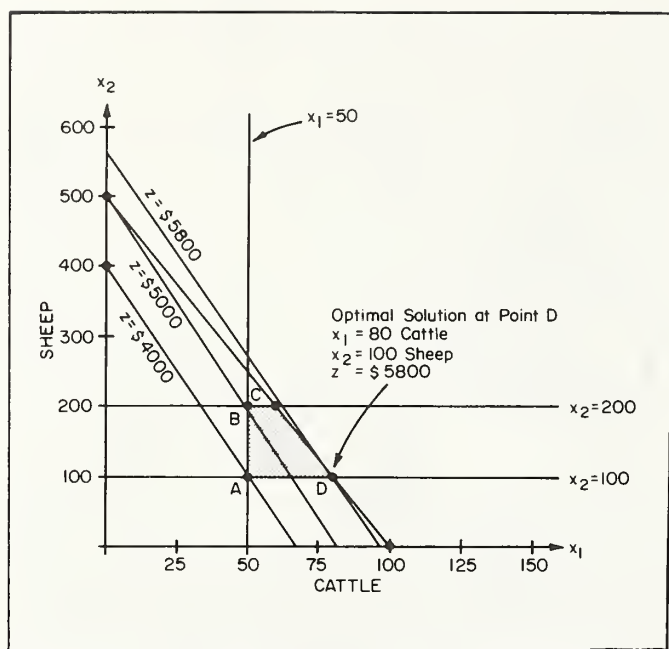


Figure 23.—The optimal solutions to the ranch problem.

## THE SIMPLEX METHOD

The trial and error graphical approach to determining the optimal solution can only be applied to two-variable problems. Fortunately, the procedure used to solve LP problems is not only relatively efficient but it works for any size problem, as long as the problem is correctly formulated (see the Advantages and Disadvantages of Linear Programming section). This method is known as the simplex method, and it is an algebraic procedure rather than a graphical one.

You may have noticed that the optimal solution of the ranch problem occurred at one of the corners of the set of feasible solutions. It can be proven mathematically that for any LP problem, the optimal solution (if it exists) will be found at one of these corners. Because of this, points located at these corners take on special importance and are known by special names, the most common of which are corner point or extreme point. The strategies or solutions that correspond to these corner points are known as *basic feasible solutions*.

**Exercise 9.**—Determine the number and location of the corner points of the set of feasible solutions for the ranch problem.

Since the optimal solution occurs at one of the corner points, only these points need to be considered in the process of determining the optimal solution.

The simplex method is a procedure for examining selected corner points, one after the other, to determine if the corresponding solution is optimal. The procedure is mechanical or repetitive in that the same computational process is used for each corner point that is examined. The underlying algebra of the method is complex, but fortunately, it is not necessary to understand the details in order to use linear programming as a tool.

For large problems, the number of corner points is quite large. Consequently, searching them for the optimal solution in a haphazard fashion would be very time consuming, even with the aid of modern, high-speed computers. Therefore, it is important that the procedure used to conduct this search is capable of finding the optimal solution as rapidly as possible.

The simplex method meets this criterion of efficiency by selecting what might be regarded as the shortest path between the starting point and the optimal solution. That is, conceptually one can regard the simplex method as moving from corner point to corner point on a journey from a starting point around the boundary of the feasible region to the optimal solution. There are usually many possible paths. Based on previous computational experience in solving large LP's, the path chosen by the simplex method is far shorter than the longest possible path.

Since, with one exception, an understanding of the algebra of the simplex method is not required, we will not describe the procedure in this report. The interested reader is referred to any of the LP texts listed at the end of the report. We make one exception to this because the initial step in the simplex method introduces some concepts that are very important in the interpretation of the solution of an LP problem.

As we have seen, LP problems involve constraints that are often in the form of  $\leq$  and/or  $\geq$  inequalities. The first step in applying the simplex method to any problem involves the conversion of all these inequalities to equations or equalities.

## Slack Variables

To see how this is done, consider the constraints in the ranch problem. The forage constraint is  $x_1 + (1/5)x_2 \leq 100$ . This constraint means that the quantity  $x_1 + (1/5)x_2$ , which is the amount of forage consumed by cattle and sheep, must be less than or equal to 100 AU's. Define a new variable  $x_3$  and set  $x_3$  equal to the difference between 100 AU's and  $x_1 + (1/5)x_2$ . That is,  $x_3 = 100 - (x_1 + (1/5)x_2)$ .

We can rewrite this equation as:

$$x_3 = 100 - x_1 - (1/5)x_2$$

or

$$x_1 + (1/5)x_2 + x_3 = 100$$

By our definition of  $x_3$ ,  $x_3 \geq 0$  since we cannot have negative forage. Our new variable is an example of what is called a slack variable in LP terminology. It measures the difference or "slack" between  $x_1 + (1/5)x_2$  and 100 AU's of forage. If  $x_1 + (1/5)x_2$  equals 100 AU's, then  $x_3 = 0$ . If  $x_1 + (1/5)x_2$  is less than 100 AU's, then  $x_3$  equals the difference between the quantities.

Note that by introducing this slack variable, we have done what we set out to do (i.e., we have converted the forage inequality into an equation). We can do the same thing for any other  $\leq$  type constraint.

**Exercise 10.**—Use the above procedure to convert the ranch hands' sheep requirement into an equation. Interpret your slack variable for this constraint.



### Surplus Variables

A similar procedure is used to convert  $\geq$  type constraints to equations. Consider the wife's sheep requirement constraint,  $x_2 \geq 100$ . This constraint states that the number of sheep ( $x_2$ ) must be greater than or equal to 100. Let us define a new variable, say  $x_4$ , that represents the difference between the actual number of sheep and 100. That is,  $x_4 = x_2 - 100$ . This may be rewritten as  $x_2 - x_4 = 100$ .

By our definition of  $x_4$ ,  $x_4 \geq 0$  since we cannot have negative sheep. The variable  $x_4$  is an example of what is called a surplus variable in LP terminology. This is because  $x_4$  measures the difference or surplus between  $x_2$  and 100 sheep. Another way to think of this is to regard all sheep in excess of 100 as extra or "surplus" sheep. For example, if  $x_2 = 100$  sheep, then there are no surplus sheep and  $x_4 = 0$ .

Note that, as with slack variables, we have converted the inequality to an equality. This procedure can be applied to any other  $\geq$  constraint.

**Exercise 11.**—Use the surplus variable procedure to convert the rancher's cattle restriction to an equality. Interpret your surplus variable for this constraint.

Prior to solving any LP problem with the simplex method, a slack variable is introduced to every  $\leq$  constraint in the problem and a surplus variable is introduced to every  $\geq$  constraint in the problem. For example, if a problem has 100  $\leq$  constraints and 78  $\geq$  constraints, there will be 100 slack variables and 78 surplus variables introduced. This is done automatically within the computer. In solving the problem, the simplex method finds values for the slack and surplus variables as well as for the original activity variables. The values of these variables provide useful information because they tell us how much of each of the resources are unused ( $\leq$  constraints) or over-produced ( $\geq$  constraints). For example, we saw that in the ranch problem the optimal solution was  $x_1 = 80$  cattle and  $x_2 = 100$  sheep (fig. 23). The amount of forage consumed is  $1(80) + 1/5(100) = 100$  AU's. That is, there is no forage left and our slack variable,  $x_3$ , is equal to zero in the optimal solution. In other words, there is no excess forage and forage is a limiting resource in this problem.

**Exercise 12.**—Determine the optimal values of the slack and surplus variables for the other three constraints in the ranch problem. Interpret the meaning of these values.

### A LAND USE PROBLEM AND SOME ELEMENTARY SENSITIVITY ANALYSIS

The three previous sections of this report have been devoted to an in-depth analysis of the ranch problem. Some linear programming (LP) terminology has been introduced and basic solution concepts have been discussed. This section will be devoted to the formulation of a simple land management problem as an LP problem. Certain aspects of the role of LP in the planning process and some complications that may arise in problem formulation will be discussed. Exercises are presented at the end of the section.

### A SIMPLE ALLOCATION PROBLEM: WILSON'S TIMBER-RECREATION MODEL<sup>3</sup>

Consider a 100,000-acre tract of national forest land located in an isolated unit. We are trying to decide how many acres to devote to timber production and how many acres to devote to concentrated recreation (campgrounds and picnic areas). That is, we are considering two management prescriptions that will correspond to the following decision variables:

$x_1$  = # of acres subjected to timber-intensive management.

$x_2$  = # of acres subjected to recreation-intensive management.

We know that the area will produce 400 board feet of timber per acre per year, on the average. No concentrated recreation developments are possible in the areas devoted to timber-intensive management because clear-cutting is the timber harvest method practiced (resulting in timber emphasis areas being completely denuded at the end of each rotation). Also, the areas are subjected to periodic commercial thinnings during the rotation. In the winter, some timber can be harvested from the areas devoted to concentrated recreation because there is no recreational use at that time. To preserve the forest environment, this cutting must be done on a selective basis and results in an average annual timber production of only 100 board feet per acre. Established timber processing plants in the vicinity have historically used 20 million board feet per year from the area. It is arbitrarily decided that the area will continue to provide at least this amount. Additional processing capacity will be encouraged up to the amount of timber that will prove to be available after the acreage allocation is determined. Timber from the area has an average end product wholesale value of \$160 per thousand board feet. Forest Service costs for timber management practices or activities average \$10 per thousand board feet produced. Net timber value is then \$150 per thousand board feet.

Campgrounds and picnic grounds are constructed to provide three units on each acre developed, but these developed areas require rather wide buffer areas and acreage for nature trails and other features. On an average, each camp or picnic unit requires 10 acres to be devoted to recreation. Each unit receives an average of 200 visits per year (or 20 visits per acre per year). There are presently 100 units on the area receiving a total of 20,000 visits per year. These visits are valued at \$5 each for what they contribute to the economy on the basis of average end product wholesale value. Forest Service recreation management costs for the area have averaged \$1 per recreation visit, leaving a net recreation value of \$4 per visit. There is a great interest in recreation, and all areas that have been developed have soon been used to capacity. Recreation planners recommend, and it is agreed, that provision for recreation visits should be at least double the present use, and that the maximum number of visits the area can sustain without serious

<sup>3</sup>Adapted from a problem presented in a Masters thesis by Carl N. Wilson, University of Montana, 1967.

degradation of recreation quality is 400,000. A summary of the cost and net value calculation is presented in table 2.

Using the information given in table 2, we will formulate a linear programming model to represent the situation. Suppose that we desire to maximize net value or revenue. Based on the information in table 2, our objective function for this problem is:

$$\text{max net revenue} = 60x_1 + 95x_2$$

or

$$\text{max } Z = 60x_1 + 95x_2$$

In order to meet the requirements outlined above, we will need the following constraints:

1. An acreage constraint (we cannot manage more than the 100,000 acres):

$$x_1 + x_2 \leq 100,000 \text{ acres.}$$

2. A timber demand constraint:

$$400x_1 + 100x_2 \geq 20,000,000 \text{ board feet.}$$

3. A minimum recreation capacity constraint:

$$20x_2 \geq 40,000 \text{ visits.}$$

4. A recreation carrying capacity constraint:

$$20x_2 \leq 400,000 \text{ visits.}$$

5. Non-negativity constraints (we cannot have a negative number of acres):

$$x_1 \geq 0, x_2 \geq 0.$$

In the event you are unsure of how the information given above was used to develop the objective function and constraints, you should review the material on units in the Introduction to Linear Programming section. Verify for yourself that the units are correct for this problem as given.

The complete LP model is:

max revenue

$$\text{Max } Z = 60x_1 + 95x_2$$

subject to:

$$x_1 + x_2 \leq 100,000$$

$$400x_1 + 100x_2 \geq 20,000,000$$

$$20x_2 \geq 40,000$$

$$20x_2 \leq 400,000$$

$$x_1 \geq 0, x_2 \geq 0$$

We can, using the terminology introduced in the General Linear Programming Model section, identify the following matrix and vectors for this problem:

1. The vector of objective function coefficients or price vector is:

$$[60, 95].$$

2. The vector of decision variables is:

$$[x_1, x_2].$$

3. The matrix of coefficients of activity variables on the LHS of the constraints (the A matrix) is:

$$A = \begin{bmatrix} 1 & 1 \\ 400 & 100 \\ 0 & 20 \\ 0 & 20 \end{bmatrix}$$

4. The vector of constraint RHS (the B vector) is:

$$B = \begin{bmatrix} 100,000 \\ 20,000,000 \\ 40,000 \\ 400,000 \end{bmatrix}$$

Table 2.—Calculation of costs and net values per acre per year.

Resource emphasis	Timber	Recreation
Timber		
Volume	400	100
Net value	$\$150 \times .4 = \$60$	$\$150 \times 0.1 = \$15$
Recreation		
Visits	0	20
Net value	0	$\$4 \times 20 = \$80$
Total net value	\$60	\$95
Total cost	\$ 4	\$21

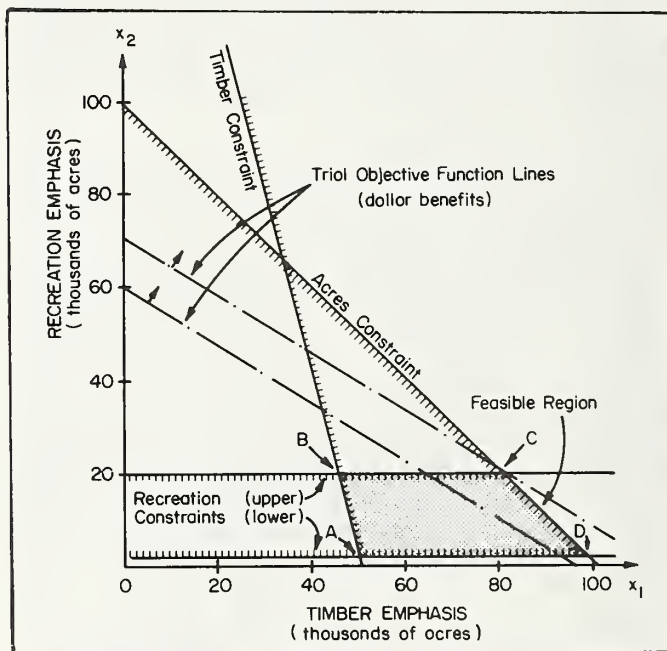


Figure 24.—Graphical representation of Wilson's TR model.

By following the steps outlined in the Graphical Interpretation of the "Stocking a Ranch" Problem section, the set of feasible solutions for the TR problem may be plotted. This plot is shown in figure 24. From the remarks we made in that section, we know the optimal solution must be located at one of the four corners or extreme points (labeled A, B, C, and D in fig. 24). Using either the graphical procedure (see fig. 24) or the simplex method, it can be shown that the optimal solution is located at point C. This solution is:

$$x_1 = 80,000 \text{ acres managed for timber;}$$

$$x_2 = 20,000 \text{ acres managed for recreation; and}$$

Max  $Z = \$6,700,000$ , of which \$4,800,000 comes from timber and \$1,900,000 comes from recreation. Verify for yourself that these objective function values are correct given the values of  $x_1$  and  $x_2$ .

## WILSON'S TR MODEL AND THE NFMA PLANNING PROCESS

### Management Practices and Management Prescriptions

Certain components of Wilson's TR problem can be related to some of the concepts or terms that have devel-



oped within the context of the NFMA planning process. For example, consider the terms *management practice*, *management prescription*, *capability area*, and *analysis area*. A management practice is defined in the regulations as:

a specific action, measure, or treatment.

A management prescription is defined in the regulations as:

management practices selected and scheduled for application on a specific area to attain multiple use and other goals and objectives.

Management prescriptions correspond to columns or decision variables in an LP allocation model.<sup>4</sup> Thus, Wilson's TR problem, timber-intensive management, and recreation-intensive management are management prescriptions. Note that each is comprised of several management practices. For example, practices included in the recreation-intensive prescription include the construction of picnic grounds and campgrounds and the selection cutting of timber.

The allocation component of an LP model developed for a forest planning effort will contain decision variables that correspond to the number of acres managed under possible prescriptions that may be applied to the land within the forest. It is important to recognize that in developing management prescriptions enough detail must be specified so that production coefficients, costs, etc., can be developed, as was done in the Wilson TR example. This is necessary because these values are an important part of the LP model.

### Analysis Areas

Within such a model, the land base for a national forest will be represented by a number of components called analysis areas and a set of prescriptions will be developed for each area. In Wilson's TR problem, there is only one analysis area (the 100,000 acre tract of land mentioned in the problem description) and only one set of prescriptions. Within the allocation model, the analysis area is the fundamental unit of land in that it is the land unit to which an acreage constraint is related. That is, there will be a constraint that ensures that the total acres allocated to all prescriptions possible for a given analysis area must be less than or equal to the total number of acres in the analysis area. For the single analysis area and set of two prescriptions in Wilson's TR problem, the acreage constraint is  $x_1 + x_2 \leq 100,000$  as given in the problem description.

As mentioned above, many analysis areas would be required in an LP model for an entire national forest. At least one acreage constraint would be necessary for each of them, and in some cases, more than one might be required. To see why this might happen suppose that for some reason it is desired in Wilson's TR problem to restrict the number of acres managed by timber-intensive management ( $x_1$ ) to be no more than 30,000; this can be

handled within the model with another acreage constraint involving only  $x_1$ . That is:

$$x_1 < 30,000$$

Thus, if it is desired to restrict the number of acres a given type of management may be applied to within an analysis area, additional acreage constraints (sometimes called secondary acreage constraints) are required. The set of acreage constraints for the different analysis areas would be one component of a forest-wide LP model.

Each analysis area has the potential of being made up of a single capability area, part of a capability area, part of several capability areas, or several capability areas. Several procedures for the identification of capability areas exist depending on factors such as the nature and type of data available, the nature of the land area in question, and the types of issues, concerns, and opportunities that must be addressed.

It is possible that several thousand capability areas might be identified on a national forest. If these were to be the fundamental units of land in an LP, several thousand acreage constraints of the type just discussed would be required. Clearly, this gets into model size problems and this is one reason why it is necessary to aggregate capability areas into analysis areas.

### Limitations of an LP Allocation Model

At this point, some comments are in order about the relationship between the solution of an LP allocation model and the problem of linking this solution to what is actually on the ground. A solution will provide information on how many acres should be managed with each prescription within each analysis area. For example, the solution of Wilson's TR problem tells us to allocate 80,000 acres to timber-intensive management and 20,000 acres to recreation-intensive management. The problem arises from the fact that the model does not tell us which acres of the 100,000 available to allocate to each prescription. This is a problem with any LP allocation model in that decisions will need to be made outside of the model as to which acres within each analysis area are managed by which prescription. In other words, each solution must be tested for spatial feasibility on the ground to ensure that it will be workable.

### The NFMA Planning Process

While Wilson's TR model is a very simple example, the steps required for its development can be related to various actions in the NFMA planning process. For example, refer to table 1 which outlines some of the planning activities that should take place during the implementation of the actions of the process. The existence of demand for timber and concentrated recreation would be determined through various planning activities within the "Identification of Issues, Concerns, and Opportunities" action. Actual demand estimates would be developed during the "Demand Analysis" phase of the "Analysis of the Management Situation" (table 1). The data on timber and recreation management

<sup>4</sup>It should be recognized that while only allocation is considered here, scheduling of prescriptions over time will also be handled through LP.



values and costs would be developed during the "Inventory Data and Information Collection" action.

The estimates of board foot yields and current recreation use would be determined in the development of resource output coefficients during the "Analysis of the Management Situation." The timber and recreation management practices, including such things as harvest systems and sizes and capacities of picnic and campsite areas, would be dealt with during the "Management prescription development" phase of the "Analysis of the Management Situation." This development would be based on findings from the "Identification of Issues, Concerns, and Opportunities" actions and "Suitability/capability analysis" activity in the "Analysis of the Management Situation." All of this information would be incorporated into the model during the LP matrix generation activities of the "Analysis of the Management Situation."

### GENERATING ALTERNATIVE PLANS BY CHANGING THE MODEL

The solution of the model yields a management plan that allocates acres to management prescriptions and produces timber and concentrated recreation. Alternative plans could be generated by making changes in the model (i.e., adding constraints, changing coefficients or the objective function, etc.) and solving the modified problems. Each solution would correspond to an alternative plan, and this would be done during both the "Analysis of the Management Situation" and the "Formulation of Alternative" actions.

#### Changing the Model to Perform a Sensitivity Analysis

**Example 16.**—With reference to Wilson's TR problem, suppose that budget is an additional concern. That is, there is a limit of \$500,000 per year that may be spent on management prescriptions.

Let us modify the original problem to incorporate this new restriction. We know from the information in table 2 that it costs \$4.00 per acre (400 board feet/acre x \$10 cost/thousand board feet) to manage for timber. For recreation management, the cost is \$21 per acre (20 units/acre x \$1/visit + 100 board feet timber/acre x \$10 cost/thousand board feet). Since we must spend less than the budget limit of \$500,000, the budget constraint may be written:

$$4x_1 + 21x_2 \leq 500,000$$

The new model is developed from the original by adding this new constraint. That is:

$$\text{Max } Z = 60x_1 + 95x_2$$

subject to:

$$\begin{array}{l} \text{acreage} \\ \text{constraint} \end{array} \quad x_1 + x_2 \leq 100,000$$

$$\begin{array}{l} \text{timber} \\ \text{demand} \\ \text{constraint} \end{array} \quad 400x_1 + 100x_2 \geq 20,000,000$$

$$\begin{array}{ll} \text{minimum} & \\ \text{recreation} & \\ \text{constraint} & 20x_2 \geq 40,000 \\ \\ \text{maximum} & \\ \text{recreation} & \\ \text{constraint} & 20x_2 \leq 400,000 \\ \\ \text{budget} & \\ \text{constraint} & 4x_1 + 21x_2 \leq 500,000 \\ \\ \text{non-} & \\ \text{negativity} & \\ \text{constraints} & x_1 \geq 0, x_2 \geq 0 \end{array}$$

At this point, it is reasonable to ask, "How does the addition of the budget constraint affect the problem in terms of selection of an optimal solution or management plan?"

To answer this question consider the graphical representation of the modified problem shown in figure 25. This shows that the addition of the budget constraint has reduced the size of the set of feasible solutions. To see this, compare the feasible region in figure 24 for the original problem with that in figure 25 for the modified problem. The stippled area contains all the points representing solutions that were feasible for the modified problem. Comparison of these figures shows that the optimal solution to the original problem (point C in fig. 24) is not feasible for the modified problem. To determine the new optimal solution we can use the graphical procedure as before. Doing this, we find that the new optimal solution is located at point  $C_1$  (fig. 25). This solution is:

$x_1 = 94,117.65$  acres managed for timber;

$x_2 = 5,882.35$  acres managed for recreation; and

Max  $Z = \$6,205,882.25$ , of which \$5,647,059.00 comes from timber and \$558,823.25 comes from recreation.

The effects of adding this budget constraint to the problem are shown in table 3.

To summarize, 14,117.65 acres were taken out of recreation production and placed in timber production in the modified problem. This resulted in a reduction of revenue of \$494,117.75 because each of the 14,117.65 acres yielded \$95 of revenue per acre when managed for recreation, but only \$60 per acre when managed for timber, a reduction of \$35 per acre.

$$(\$35)(14,117.65) = \$494,117.75$$

This reduction was caused by the fact that we could only spend \$500,000 because of our budget limitation; and in the original problem, where budget was unconstrained, we spent  $(\$4)(80,000) + (\$21)(20,000) = \$740,000$ .

To meet the required budget reduction, acres managed for recreation in the original plan were shifted to timber management because of the lower cost.

As mentioned above, each time we modify and solve any LP planning model, we generate an alternative plan. A major purpose in doing this is to be able to analyze trade-offs between alternative plans (Analysis of the Management Situation and Formulation of Alternative actions, table 1). In the example just cited, we have a potential trade-off between budget and revenue. A reduc-

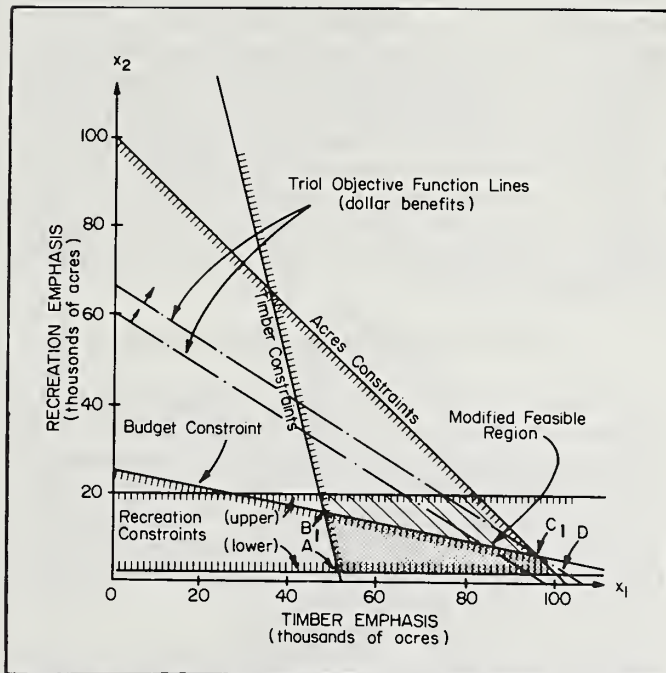


Figure 25.—Graphical representation of Wilson's TR model with budget constraint added.

tion in our budget from \$740,000 to \$500,000 leads to a reduction in revenue of \$494,117.65. Any comparison of these two plans would need to consider this trade-off. If this were a real situation, it would be necessary to determine if an increase in revenue of almost \$500,000 justifies an increased expenditure of \$240,000. In the same way, other trade-offs arising from changes in the model can be analyzed (see exercise 14).

We have just described a very simple example where we have investigated the effects of a change in the model on the optimal solution. This sort of analysis is often called a *sensitivity analysis* or *post-optimality analysis*. For a small problem like this, the effect of such a change is easy to see since we can use graphics to illustrate it. However, for a large problem, the effect of such a change can be quite subtle and a more careful analysis is needed. Most computer packages print out, in addition to the optimal solution, information that aids in sensitivity analysis.

One important type of change in an LP model that will be made in order to generate alternative plans is a change of objective functions. Within a given LP model, one can choose any row or relationship to be the objective function. For example, consider the modified Wilson TR

problem developed in example 17. The objective function chosen is the maximization of net revenue. Other objective functions that could be chosen are:

1. Maximize timber production;
2. Minimize timber production;
3. Maximize recreation production;
4. Minimize recreation production;
5. Minimize budget.

It should be obvious that choosing one of these alternative objective functions will often lead to a different optimal solution than the one corresponding to the corner point  $C_1$  (fig. 25). If we assume that all constraints are as specified in example 16, then the feasible region will be as shown in figure 25. From what we have said about the simplex method, we know that the optimal solution, regardless of our choice of objective functions must occur at one of the four corner points A,  $B_1$ ,  $C_1$ , or D. Of course, each of these corresponds to different allocations of the land and, hence, different alternative plans.

#### Altering the Objective Function— Maximize Timber Production

**Example 17.**—Consider the objective function "Maximize Timber Production." That is:

$$\text{Max } Z = 400x_1 + 100x_2$$

subject to:

$$\begin{array}{ll} \text{acreage} & \\ \text{constraint} & x_1 + x_2 \leq 100,000 \end{array}$$

$$\begin{array}{ll} \text{timber} & \\ \text{demand} & \\ \text{constraint} & 400x_1 + 100x_2 \geq 20,000,000 \end{array}$$

$$\begin{array}{ll} \text{minimum} & \\ \text{recreation} & \\ \text{constraint} & 20x_2 \geq 40,000 \end{array}$$

$$\begin{array}{ll} \text{maximum} & \\ \text{recreation} & \\ \text{constraint} & 20x_2 \leq 400,000 \end{array}$$

$$\begin{array}{ll} \text{budget} & \\ \text{constraint} & 4x_1 + 21x_2 \leq 500,000 \end{array}$$

$$\begin{array}{ll} \text{non-} & \\ \text{negativity} & \\ \text{constraints} & x_1 \geq 0, x_2 \geq 0 \end{array}$$

In order to maximize timber production, as many acres as is feasible should be allocated to the timber intensive prescription. Since 2,000 acres must be allocated

Table 3.—Comparison of optimal solutions for two versions of Wilson's TR model

Item	Original solution	New solution	Amount of change	Direction of change
Timber ( $x_1$ ) (acres)	80,000	94,117.65	14,117.65	increase
Recreation ( $x_2$ ) (acres)	20,000	5,882.35	14,117.65	decrease
Revenue (Z)	\$6,700,000	\$6,205,882.35	\$494,117.65	decrease



to the recreation prescription in order to meet the minimum recreation constraint, 98,000 acres are available for timber intensive management and the optimal solution occurs at the point D,  $x_1 = 98,000$  acres,  $x_2 = 2000$  acres (fig. 25). From this, it is easy to determine the corresponding levels of outputs of the various products and services as was done in the previous examples.

### Altering the Objective Function—Minimize Budget

**Example 18.**—Consider the objective function “Minimize Budget.” That is:

$$\min Z = 4x_1 + 21x_2$$

subject to the same constraints as shown in example 17.

Clearly, the way to minimize budget is to minimize the number of acres managed, especially the number of acres managed under the recreation prescription since it is more costly (\$21 versus \$4 for the timber prescription).

The corner point A corresponds to this, and the solution is to manage the minimum of 2,000 acres under the recreation prescription and 49,500 acres under the timber prescription. Note that in this solution, 48,500 acres are not managed by either prescription. If it is required that all acres be managed in some fashion, this would be an unacceptable solution. There are at least two ways in which the problem could be modified in order to resolve this difficulty. The simplest change would involve the modification of the acreage constraint to force all of the acres to be managed. That is:

$$x_1 + x_2 = 100,000$$

Note that if this were done, the feasible region would become the line segment  $C_1D$  in figure 25 (see exercise 13). It is possible that forcing all acres to be managed under one or the other of these prescriptions may be undesirable. An additional prescription, a minimum level of management, could be defined. If the number of acres to be allocated to this prescription is denoted by  $x_3$ , then the acreage constraint would be:

$$x_1 + x_2 + x_3 = 100,000$$

If no other changes are made in the model, then the 48,500 acres not managed under the timber and recreation prescriptions would be managed under the minimum level of management in this example.

One final comment on examples 17 and 18 is in order. As presented, there is no consideration of net revenue in the model once the objective function is changed. It would be easy to rectify this by adding a revenue constraint if this were desired. For example, if it were desired to generate at least \$5,000,000 in net revenue, a constraint of the form:

$$60x_1 + 95x_2 \geq 5,000,000$$

could be included in the model.

### SOME ADDITIONAL ASPECTS OF LP PROBLEM FORMULATION

Wilson's Timber-Recreation model illustrates problems that may occur during LP problem formulation. These

difficulties can be encountered (and often are) in LP problems of any size, and their characteristics are easy to demonstrate in a two-variable problem because graphical representation is possible. Some aspects of problem formulation such as an infinite number of optimal solutions and redundant constraints do not cause any difficulties in determining an optimal solution, but others such as non-feasible solutions or an unbounded optimal solution make determination of an optimal solution impossible.

### Infinite Number of Optimal Solutions

Until now the LP problems considered have had a unique optimal solution. However, it is possible for an infinite number of solutions satisfying the criterion of optimality to exist; that is, each of these solutions maximize or minimize the objective function. To see how this can happen we will change the objective function for the Timber-Recreation problem by changing the coefficient of  $x_2$  from \$95 to \$60. That is:

$$\max \text{revenue } (Z) = 60x_1 + 60x_2$$

This is, of course, equivalent to changing the net return for intensive recreation management from \$95/acre to \$60/acre.

If we consider the original problem (no budget constraint), then the set of feasible solutions shown in figure 26 is exactly the same as that shown in figure 24. However, when we use the graphical approach to solve the problem with the modified objective function, we note that the slope of the lines representing trial objective function values has changed because of the change in the coefficient of  $x_2$ . In fact, the slope is the same as that of the line representing the acreage constraint. As we plot lines corresponding to increasing values of revenue, we find that eventually (at revenue = \$600,000) the objective function line and the acreage constraint line are one and the same.

Recall that in our previous examples, as we selected increasingly large trial values for the objective function, we eventually reached a value where the objective function line intersected only one point (a corner point or extreme point) in the set of feasible solutions. This point corresponded to the optimal solution (assuming a maximization problem) because when we picked larger values for the objective function, the corresponding lines lay completely outside the set of feasible solutions. That is, these lines correspond to values of profit or revenue that are larger than the values yielded by any feasible solution.

This holds for the example in figure 26 in that selection of any objective function value greater than \$600,000 leads to a line that lies outside the feasible region. The important point to note is that this objective function line intersects the feasible region not at just one point, but rather at every point on the line segment  $C_1D$ . This means that every point on this line segment corresponds to an optimal solution because all these points yield a \$600,000 return. This is easy to verify by determining the coordinates of any point on this line segment and substituting them into the objective function. For exam-



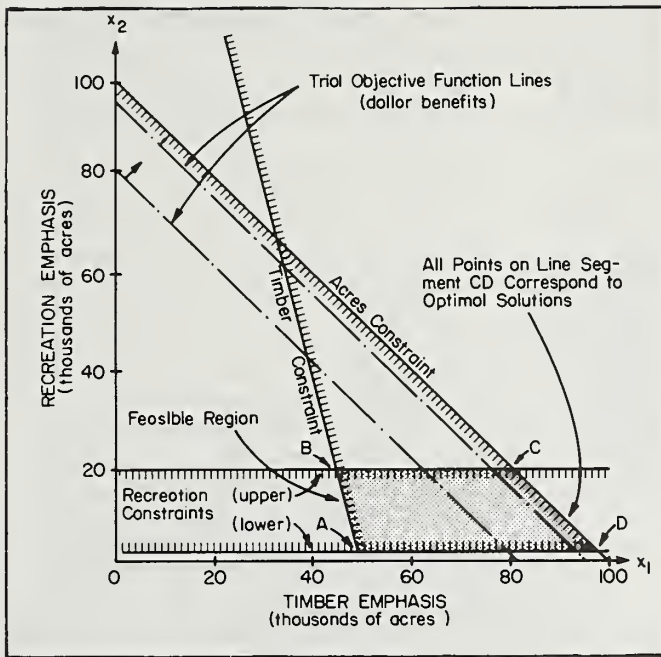


Figure 26.—Multiple optimal solutions encountered when objective function is changed to  $Z = 60x_1 + 60x_2$ .

ple, the coordinates of the point D are  $x_1 = 98,000$ ,  $x_2 = 2,000$ , and:

$$\text{return} = (60)(98,000) + (60)(2,000) = \$600,000.$$

Hence, any solution (there are an infinite number of these) on this line segment is optimal and may be implemented. Note that this is caused by the fact that the slope of the objective function line is the same as the slope of one of the constraint lines (in this case the acreage constraint). While this can occur with problems of any size, it does not present any difficulties in using the simplex method to determine the optimal solution. In fact, the simplex method will detect multiple optimal solutions if they exist.

### Redundant Constraints

Consider the addition of a budget constraint to the original TR model, but now assume that the budget limitation is \$750,000 instead of \$500,000. That is, the budget constraint is:

$$4x_1 + 21x_2 \geq 750,000$$

A graphical representation of the problem including this constraint is presented in figure 27. Note that this constraint lies outside the set of feasible solutions but that every point in the set of feasible solutions satisfies it since it is a  $\leq$  constraint. Thus, it does not alter the original set of feasible solutions, and the optimal solution would be exactly the same as for the original problem. A constraint that does not alter the problem by changing the set of feasible solutions is called a redundant constraint. In large problems, redundant constraints are difficult to detect because a graphical representation of the problem cannot be developed. Most efficient computer software LP packages eliminate redundant constraints for the sake of efficiency of solution before attempting to solve the problem.

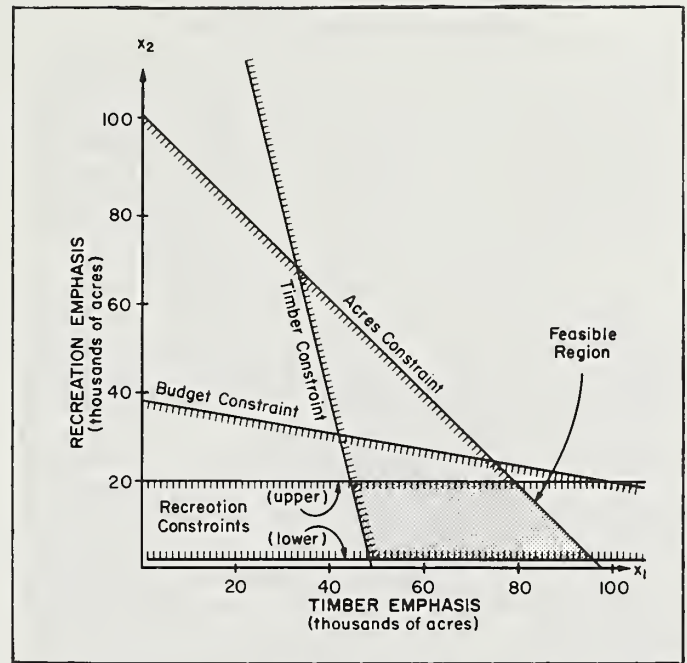


Figure 27.—Graphical representation of TR problem with budget limit of \$750,000.

Another way to describe redundant constraints is to say they are not binding or that other constraints in the problem are more binding or more constraining. This is true here because if you will recall, we spent \$740,000 to implement the optimal strategy for the original TR problem. This was the most money we could spend because of the acreage constraint; that is, we had only 100,000 acres to manage. Since we can spend at most \$740,000, it is obvious that an extra \$10,000 (a budget of \$750,000) is not going to do us any good. In other words, the acreage limit is more binding than a budget of \$750,000 because we cannot spend that much money on the available acreage given the other constraints. Of course, if some other constraint were changed, then it might be possible to exceed a \$750,000 budget and, thus, make this constraint binding (see exercise 15).

### No Feasible Solution

To illustrate this situation consider a budget limitation of \$160,000. That is, the budget constraint is:

$$4x_1 + 21x_2 \leq 160,000$$

A graphical representation of this situation is shown in figure 28.

Note that since the budget constraint is a  $\leq$  type constraint, the only points which represent solutions that require less than \$160,000 to implement lay below the budget constraint line. Therefore, none of the solutions that satisfy the budget constraint also satisfy the timber constraint. As a result, there are no solutions that simultaneously satisfy these constraints and there is no set of feasible solutions to this problem. Problems like this cannot be solved without reformulation to eliminate the infeasibility and this is possible only if the system being modified has a set of feasible solutions in the first place.

## Unbounded Optimal Solution

Now we will modify the TR problem so that the optimal solution is unbounded. That is, at least one of the decision variables can take on arbitrarily large values and still be feasible. This situation occurs if we eliminate the acreage constraint from the original problem. A graphical representation of this situation is presented in figure 29.

Note that  $x_1$  (acres managed for timber) can take on any value up to infinity. Since  $x_1$  is now unbounded, it is possible to manage an unlimited number of acres for timber and generate unlimited net return. This is obviously an unrealistic situation and no realistic solution to the problem can be determined.

Because we have dealt only with small problems, it may seem that the conditions leading to no feasible solution or an unbounded optimal solution are easy to detect. For real world problems, however, such conditions are often difficult to identify. Most computer LP packages provide some clues as to the cause of an infeasibility or an unbounded solution, but these problems can be difficult to locate and solve even with this additional information. When large scale LP models are formulated, a great deal of time is often devoted to locating and correcting problems causing infeasibilities. It is important to realize that if the model is correctly formulated (i.e., if it is an accurate representation of the system), then an unbounded solution should not be present.

The following exercises refer to the TR model discussed above.

**Exercise 13.**—With reference to the original TR problem, suppose the acreage constraint is changed to:

$$x_1 + x_2 = 100,000 \text{ acres}$$

How would this modify the set of feasible solutions for

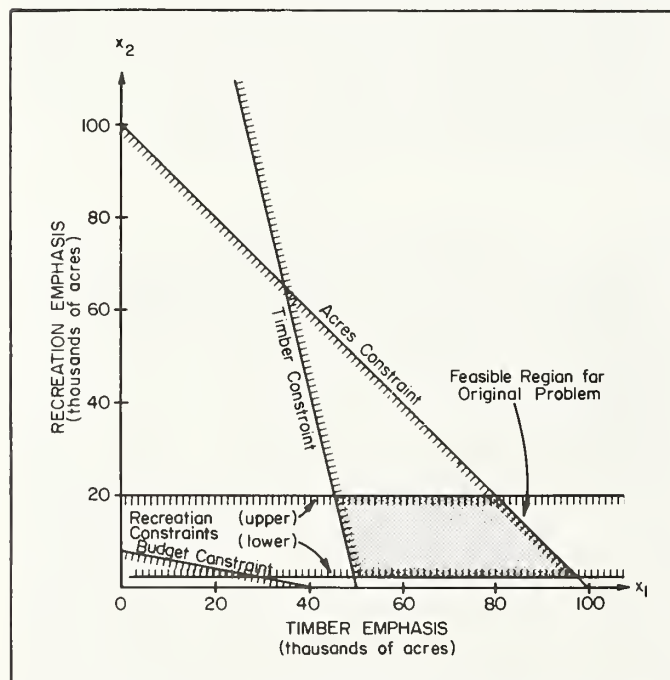


Figure 28.—Graphical representation of TR problem with budget limit of \$160,000.

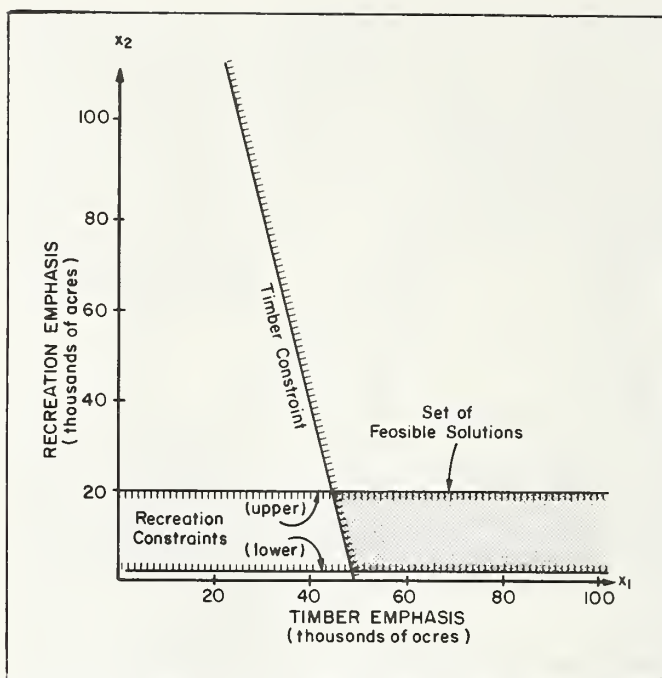


Figure 29.—Graphical representation of TR problem with no acreage constraint.

the problem? Graph this new set. From a management standpoint, what is the implication of this change in the acreage constraint? Will the optimal solution for the problem change?

**Exercise 14.**—Consider the modified TR problem with the budget constraint:

$$4x_1 + 21x_2 \leq 500,000$$

as discussed in the text. We have identified the trade-offs in budget and revenue that result from the imposition of this constraint. Discuss all additional trade-offs that result from this new constraint. Be sure to calculate the actual amounts involved in each trade-off.

**Exercise 15.**—With reference to figure 27, reformulate the TR problem so that the budget constraint:

$$4x_1 + 21x_2 \leq 750,000$$

is no longer redundant. How many different constraints could be changed (one at a time) to accomplish this?

**Exercise 16.**—With reference to figure 28, reformulate the TR problem with the budget constraint:

$$4x_1 + 21x_2 \leq 160,000$$

so that there is a set of feasible solutions to the problem. Graph this reformulated problem, identify the set of feasible solutions, and determine the optimal solution. Do you have any redundant constraints in your reformulated problem?

**Exercise 17.**—Consider the original form of the TR model. Suppose we want to change the criterion of optimality to the minimization of budget expenditures. Also, suppose that we require that net returns must be at least \$400,000. All other requirements are as



given in the original problem. Formulate this new problem, write out the A matrix, B vector, draw a graphical representation of the problem, label the set of feasible solutions, and determine the optimal solution to the problem.

The following exercises deal with additional land management problems.

**Exercise 18.**—Curly Fuzz has 800 acres of land which is suitable for either camping or skiing. He also has allocated 100 hours of his own time per month to operate the enterprises. The camping enterprise can be operated 7 months of the year while skiing can be conducted for only 5 months. Each acre devoted to camping requires 1 man hour per year for operation and returns \$20 net income. Skiing requires the same labor per acre but returns \$30 net income per acre. What is the best mix of acres to devote to camping and skiing to maximize Fuzz's net income if a given acre can be used for one and only one enterprise (Duerr et al. 1975)?

Formulate this as a linear programming problem, draw a graphical representation of the problem, and determine the optimal solution.

**Exercise 19.**—Suppose you have 300 acres which may be regenerated by one of two possible methods, machine planting or aerial seeding. Each acre planted contributes \$25 to your profit and each acre seeded contributes \$10. Two man hours are expended in planting an acre while only one man hour is required to seed an acre. Lastly, machine planting an acre requires one planter machine hour while aerial seeding an acre requires one-fourth hour flying time. At most you can plan on 400 man hours, 200 planter machine hours, and 60 flying hours for the entire project. The objective is to maximize profits.

1. Formulate this as an LP problem.
2. Define all variables.
3. Draw a graphical representation of the problem.
4. Determine the optimal solution.

**Exercise 20.**—The Big'r-N-Hell Timber Company is bidding on a 100-acre block of timber on national forest land and wishes to know what revenue can be obtained from the harvest before bidding. The Forest Service has stated that at least 100 acre feet of water must flow down the stream after the cutting operation and that sediment production cannot increase by more than 40 tons. The firm can clearcut or cut selectively; the first method yields a profit of \$500 per acre and the second yields \$75 per acre. Clearcutting an acre produces 3 acre feet of water and increases sediment 0.5 ton while selective cutting produces 0.5 acre feet of water and increases sediment 0.1 ton. Set up a linear program that will maximize profit from timber harvesting on the 100 acres. Solve this graphically.

The following problems deal with complications arising from improper formulation. To answer them draw the graphical representation for each problem.

**Exercise 21.**—Can the following problems be solved as an LP problem? State why or why not.

$$\begin{aligned} \max Z &= 5x_1 + 10x_2 \\ \text{subject to: } &5x_1 - x_2 = 10 \end{aligned}$$

$$2x_1 - x_2 = 20$$

**Exercise 22.**—Can the following LP problem be solved? State why or why not.

$$\begin{aligned} \max Z &= 5x_1 + 6x_2 \\ \text{subject to: } &10x_1 + 10x_2 \leq 100 \\ &x_2 = 6 \end{aligned}$$

## ADVANTAGES AND DISADVANTAGES OF LINEAR PROGRAMMING

As with any technique, Linear Programming (LP) has certain advantages and disadvantages which influence or limit its applicability. In this section, we will discuss the assumptions internal to linear programming and their influence on application of LP to land management problems. We will also compare LP with some other quantitative modeling techniques that may serve as alternatives.

### ASSUMPTIONS OF LINEAR PROGRAMMING

The assumptions discussed include (1) linearity, (2) proportionality, (3) additivity, (4) non-negativity, (5) divisibility, (6) certainty, and (7) a single objective function.

#### Linearity

All mathematical relationships expressed in any LP model, either in the objective function or the constraints, must be linear in terms of the decision variables. That is, there cannot be any terms involving powers, cross-products, logarithms, etc. For example, the expressions:

$$\begin{aligned} \max Z &= 16x_1 + 2x_2 + 4x_3 \\ 4x_1 + 5x_2 + 6x_3 &\leq 100 \\ x_1 + 9x_2 + 11x_3 &\geq 50 \end{aligned}$$

are all linear, but the expressions:

$$\begin{aligned} \max Z &= 16x_1^2 + 4x_1x_2 + 9x_3 \\ 19x_1^2 + 4x_1x_2 + 6 \log x_3 &\leq 100 \\ 4x_1^2 + 11x_2^4 + 4x_1x_2x_3 &\geq 50 \end{aligned}$$

are not linear. The last three expressions would be invalid expressions for inclusion in an LP model. They could, however, be incorporated in certain nonlinear programming models.

In order to satisfy the linearity requirement, two additional assumptions about relationships between decision variables are implied. These are known as proportionality and additivity and they must hold in the objective function and in all constraints.

#### Proportionality

Proportionality means that the measure of effectiveness (objective function) and rate of resource usage (constraints) must be proportional to the level of each activity conducted separately. For example, suppose that the objective function of interest involves profit maximization and that  $x_1$  represents the number of acres on which a particular management prescription is applied. If the coefficient of  $x_1$  in the objective function is \$50, this means that each acre of land managed under the ap-

appropriate prescription will yield a profit of \$50. That is:

if  $x_1 = 1$ , profit is  $\$50(1) = \$50$ ;

if  $x_1 = 49$ , profit is  $\$50(49) = \$2,450$ ;

if  $x_1 = 50$ , profit is  $\$50(50) = \$2,500$ .

Said another way, no matter how many acres are currently managed under this prescription ( $x_1$ ), the rate of increase (decrease) in profit from this activity will be a \$50 per acre increase (decrease) in the number of acres managed.

Proportionality implies the same relationship for the variables and their coefficients in the constraints except that now the coefficients are a measure of rate of resource use rather than a measure of effectiveness. For example, suppose we have a constraint of the form:

$$5x_1 + 10x_2 + 6x_3 \leq 4,000$$

where 4,000 is the number of units of some resource that is available and, again,  $x_1$  is the number of acres to be managed according to some management prescription. Proportionality means that no matter how many acres we manage under this prescription, each acre will consume 5 units of the resource.

### Additivity

The assumption of proportionality alone is not enough to guarantee linearity; it is possible for interactions between activities to occur (represented mathematically by cross-product terms) with respect to total effectiveness or usage. To insure that this cannot happen, the assumption of additivity must be made. That is, the total effectiveness (objective function) and total resource usage (constraints) resulting from conducting two or more prescriptions at the same time must equal the sums of effectiveness or usage resulting from each activity being carried out separately.

For example, suppose the objective function is:

$$\text{max profit} = 50x_1 + 30x_2 + 40x_3$$

and  $x_1$ ,  $x_2$ , and  $x_3$  equal the number of acres managed under each of three different prescriptions. The coefficients represent the per acre profits for each. Further suppose that:

$x_1 = 20$  acres managed, prescription 1;

$x_2 = 30$  acres managed, prescription 2;

$x_3 = 40$  acres managed, prescription 3.

The assumption of proportionality guarantees that:

profit from prescription 1 =  $(50)(20) = \$1,000$ ;

profit from prescription 2 =  $(30)(30) = \$900$ ;

profit from prescription 3 =  $(40)(40) = \$1,600$ .

The assumption of additivity guarantees that the total profit from all three prescriptions will equal the sum of the profits resulting from each one separately. In this case:

$$\text{profit} = \$1,000 + \$900 + \$1,600 = \$3,500$$

The assumption of additivity implies the same thing with respect to rates of consumption of resources as expressed by constraints. That is, if our constraint is:

$$5x_1 + 10x_2 + 6x_3 \leq 4,000$$

and  $x_1$ ,  $x_2$ , and  $x_3$  have the values given above, then the assumption of proportionality guarantees that:

units of resource used by prescription 1 is  $(5)(20) = 100$ ;

units of resource used by prescription 2 is  $(10)(30) = 300$ ; and

units of resource used by prescription 3 is  $(6)(40) = 240$ . The assumption of additivity guarantees that the total amount of resource used is the sum of amounts used by each prescription carried out separately. In this case, amount of resource used =  $100 + 300 + 240 = 640$  units.

### Non-Negativity

As has been mentioned, it is assumed that all decision variables are non-negative or that they cannot take on negative values. This assumption is rarely violated in practice, particularly in land management applications. Even if it is violated, it presents no problem in solving the LP problems because of various tricks that can be applied.

### Divisibility

This assumption refers to the ability of the decision variables to take on any non-negative value rather than just integer values. In other words, the decision variables are assumed to be continuous variables.

### Certainty

It is assumed that all coefficients in the model (i.e., both those in the objective function and those in the constraints) are known constants. That is, an LP model is a deterministic model or one that involves no uncertainties or probabilistic elements.

### A Single Objective Function

Linear programming is often criticized because it considers only one criterion of optimality (objective function) at a time. It is possible to optimize with respect to more than one criterion, but each must be considered separately. This can be done by repeatedly using the simplex method to determine the optimal solution corresponding to each objective function of interest. Simultaneous optimization for two or more criteria is possible only if they can be expressed in the same units and, thus, be incorporated in a single objective function. Criteria or objectives that cannot be expressed in the same units are often referred to as incommensurable objectives and simultaneous optimization of two or more of these is beyond the capability of any technique.

### ADVANTAGES AND DISADVANTAGES OF LINEAR PROGRAMMING

Like all analytical techniques, the assumptions on which LP is based create advantages and disadvantages that influence its application. It is important to be aware of these and to recognize the limitations they imply. Linear programming has often been criticized for its disadvantages (i.e., linearity, lack of certain knowledge about the coefficients, lack of enough detail to adequately reflect reality, etc.), and while it is certainly important



to recognize these, it is just as important to realize that the alternatives to LP also have their disadvantages. Furthermore, LP possesses certain advantages that place it in a favorable position when compared to some of the alternative modeling techniques.

### The Linearity Assumption: A Discussion

Perhaps the linearity assumption of LP has been the target of more criticism than any other aspect of the technique. While it is certainly true that the world is not linear, this restriction in an LP model may not be as serious a problem as some claim it is. One reason for this is that many nonlinear relationships can be approximated by linear ones. This is often done in a piecewise manner as is indicated by the examples in figure 30.

From these examples, it can be seen that one can closely approximate a nonlinear relationship with segments of straight lines. In such cases, a linear approximation is often sufficient, particularly when one considers our lack of knowledge about some aspects of natural resource systems.

A special type of linear programming known as separable programming is designed to handle nonlinear constraints by using a piecewise linear approximation like that shown in figure 30. Piecewise linear approx-

imation of monotonically decreasing constraints (Case A in fig. 30) enables such relationships to be incorporated in conventional linear programming models. Such an approach has been utilized to handle downward-sloping demand curves in timber production (Hrubes and Navon 1976).

If one simply cannot live with linear approximations of nonlinear relationships, there are various nonlinear programming techniques that permit inclusion of nonlinear constraints or objective functions in the model. Unfortunately, while nonlinear programming models can be formulated, solution of the same models is often not possible. This is particularly true as model size increases. In other words, we can relax the linearity assumption and formulate a nonlinear model, but once we have done this we have no guarantee that we can solve the problem. In linear programming, if all of the assumptions are met, the optimal solution can be determined if it exists.

It also turns out that if we happen to be able to solve our nonlinear programming problem, it will cost us more than it would for an LP problem of comparable size. This is because algorithms for solving nonlinear programming problems are more complex and computationally more inefficient than the simplex method. For large scale problems like those encountered in land management planning, the difference in costs can be quite significant. The guaranteed solution and computational efficiency of the simplex method are two factors that make it advisable to sacrifice nonlinearity for large scale problems.

### The Divisibility Requirement: A Discussion

The divisibility requirement, or the assumption that the decision variables are continuous, can lead to problems. Often these variables represent quantities that have significance only at integer values. For example, it makes no sense to talk about 101.78 cattle, 2.63 campsites, or 1,050.92 deer. When this sort of thing happens, a common practice is to round down (i.e., 101 cattle, 2 campsites or 1,050 deer). This practice can be dangerous because it is possible that these rounded values are either infeasible or non-optimal. Mixed integer programming is a technique that handles problems with both continuous and discrete decision variables, but unfortunately, the solution algorithm, like those for nonlinear programming, can be inefficient and costly to use relative to the simplex method.

### LP as a Deterministic Model: A Discussion

Our limited understanding of natural resource systems poses a problem because LP is a deterministic model. It implies that there are no unknown coefficients. This can make it difficult to assess both favorable effects such as board-foot yields per acre, cow-calf AUM's, and acre feet of water produced; and unfavorable impacts such as tons of sediment resulting from implementation of management prescriptions.

Furthermore, many of the coefficients are estimates of conditions or outputs that are to take place in the

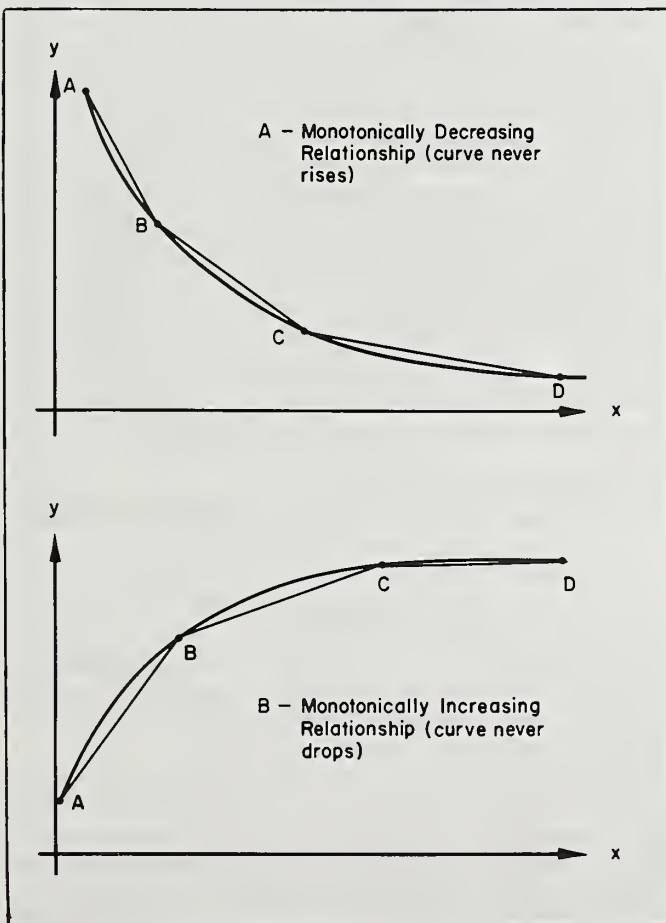


Figure 30.—Examples of piecewise linear approximation of nonlinear relationships.

future. Many land management LP models consider the scheduling of prescriptions in the future and do not consider the possibility that these activities may not occur. An LP model's allocation of acres to prescriptions is dependent, in part, on the quantities of different products that are produced since they are the measure by which the merits of each prescription are assessed. Thus, incorrect values for these coefficients can lead to a different allocation of acres and, hence, different levels of resource outputs than would be obtained if proper values were utilized.

Fortunately, the influence both of changed coefficients and of future changes in activities on the acreage allocation can be assessed through sensitivity analysis. Linear programming models with probabilistic components have been developed, but they suffer from the same disadvantages in terms of solution efficiency as nonlinear programming and mixed integer programming.

### **Nonquantifiable Benefits: A Discussion**

Related to prescriptions whose beneficial effects are not quantifiable is the need to quantify the effects of management prescriptions. This limitation influences model development in the same way that uncertainty does. Examples include the printing of brochures designed to disperse recreation use, range practices that keep cattle away from heavily grazed areas, and research into particular aspects of the biological system being managed. The solution to this problem lies in the recognition that no single model or approach can incorporate all aspects of a "real world" problem. Specifically, only those practices that can be well quantified should be included in an LP model. That is, while the LP allocation of prescriptions may provide the basis of a plan, the complete plan will also include numerous items never considered by or incorporated in the model itself. In some cases, other analytical tools such as public involvement procedures and input-output analysis may be utilized to assess the merit of prescription (Betters 1977).

### **MODEL SIZE**

Several of the above considerations relate to model size, a concept we have alluded to several times in this report. As mentioned above, one advantage of LP is that several thousand variables and mathematical relationships can be incorporated into a single model. Even so, there are problem size limitations in terms of model costs arising both from required computer solution time and storage requirements for the data included in the model. Even the most sophisticated computer LP solution packages have limitations on problem size, and in addition, output from very large problems can be difficult to interpret. As an example, the FMPS package on the system at Fort Collins cannot handle problems with more than 8,000 rows. In short, restricting the model by including only those factors that can be meaningfully quantified is an important consideration if LP is to be utilized effectively. Again, this is a reflection of the point made

above that no single model can incorporate all aspects of a "real world" problem.

In conclusion, LP, like all modeling techniques, has numerous advantages and disadvantages. Important reasons for its being recommended for forest planning include:

1. It is one of few techniques that can be used to deal with problems of the size encountered in forest planning;
2. It is an optimization technique and, thus, may be used to satisfy some of the requirements in the NFMA regulations;
3. It is more nearly operational within the agency than other modeling techniques; and
4. There is some background of experience within the agency from prior LP modeling efforts.

It should not be implied from the preceding discussion that LP is the panacea to the forest planning problem that some believe it to be.

One does not simply formulate an LP, generate some runs, and have a set of forest plan alternatives. Considerable additional work is needed to take an LP solution and use it to develop an alternative plan.

Research is needed for the purpose of developing improved modeling approaches. For example, the fact that an LP model considers only a single objective function was mentioned earlier in this section. Land management planning on national forests must respond to many conflicting objectives, and the normal practice, in cases where LP has been applied, has been to determine the optimal solution for each objective of interest. The disadvantage of this approach is that each objective or criterion of optimality is considered more or less independently when simultaneous consideration of several objectives is desirable. Unfortunately, as was mentioned above, a single solution that optimizes a set of conflicting objectives simultaneously does not exist. Multiobjective optimization techniques should be investigated in order to determine their applicability to forest planning problems. Following is a brief introduction to one multiobjective technique, goal programming, which happens to be a special case of linear programming.

### **GOAL PROGRAMMING: A SPECIAL CASE OF LINEAR PROGRAMMING**

As described above, one limitation of LP is the fact that it cannot deal with problems involving multiple incommensurable objectives. However, there is a class of techniques known as multiple optimization or vector optimization techniques that employ multiple decision criteria for the purpose of determining a preferred or satisfying solution to such problems.

The multiple optimization technique that has been applied most often is known as goal programming. While this technique is not the subject of this report, its usage is widespread enough to justify a brief introduction. The primary purpose of this introduction is to provide the reader with some insight into the relationship between the LP and goal programming (Field 1978, Field et al. 1980).



## An Example

One way to introduce goal programming is by means of an example. Toward that end consider the following modification of Wilson's TR problem (see the Land Use Problem and Some Elementary Sensitivity Analysis section) into a goal programming problem.

Suppose the land manager has identified the following goals:

1. He would like to maximize the production of recreation visits.
2. He would like to harvest as much timber as possible.

In order to represent these goals in a goal programming model, it is necessary to quantify them by specifying targets. There are several ways of doing this, but a recommended procedure is to specify targets that correspond to the optimum level for each goal or criterion considered by itself. This can be done by solving two separate LP problems, one with an objective function that will maximize the production of recreation visits and the other with an objective function that maximizes timber harvesting. Each LP would have the set of constraints defined for the original Wilson's TR problem.

If the reader reviews the discussion in the Land Use Problem section and examines the graphical representation of Wilson's TR model in figure 24, it is easy to determine that the optimal solutions for each of the LP problems correspond to the corner points C and D of the set of feasible solutions. Thus, for the first goal:

the target value = 400,000 recreation visits  
and this is obtained by managing 80,000 acres of timber ( $x_1 = 80,000$ ) and 20,000 acres for recreation ( $x_2 = 20,000$ ). For the second goal:

the target value = 39,400,000 board feet of timber  
and this is obtained by managing 98,000 acres for timber ( $x_1 = 98,000$ ) and 2,000 acres for recreation ( $x_2 = 2,000$ ).

It should be obvious to the reader that the objective functions for each of the LP problems just discussed would be:

for goal 1: max recreation visits =  $20x_2$ ;

for goal 2: max timber production =  $400x_1 + 100x_2$ .  
Let us now consider a different way of expressing the objective functions that correspond to each of these goals. In doing so, we can examine one of the differences between LP and goal programming, this being the way in which the objective function is expressed.

### Defining the First Goal

For now, consider only the first goal. The constraints for the original problem are:

$$\begin{aligned} x_1 + x_2 &\leq 100,000 \\ 400x_1 + 100x_2 &\geq 20,000,000 \\ 20x_2 &\geq 40,000 \\ 20x_2 &\leq 400,000 \\ x_1 &\geq 0, x_2 &\geq 0 \end{aligned}$$

The fourth constraint pertains to our recreation goal. In our discussion of the simplex method in the Land Use Problem and Some Elementary Sensitivity Analysis section, we introduced the concept of slack and surplus variables. Let  $x_3$  be the slack variable for this constraint.

Then we have:

$$20x_2 + x_3 = 400,000 \text{ visitor days.}$$

As we saw in the Land Use Problem section,  $x_3$  measures the difference between the amount of visitor days produced and the limit of 400,000 imposed by the carrying capacity of the system. That is:

$$x_3 = 400,000 - 20x_2$$

This suggests that we can state the manager's recreation goal in terms of the slack variable  $x_3$  because maximizing the production of recreation visits is equivalent to minimizing  $x_3$  because, as discussed above, 400,000 visitor days is both the target for the goal and the maximum amount that can be produced. The goal programming formulation for the first goal considered above is:

$$\begin{aligned} \min Z_1 &= x_3 \\ \text{subject to:} \\ x_1 + x_2 &\leq 100,000 \\ 400x_1 + 100x_2 &\geq 20,000,000 \\ 20x_2 &\geq 40,000 \\ 20x_2 + x_3 &= 400,000 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0 \end{aligned}$$

Note that this simple example of a goal programming problem is just a minimization LP problem with the objective function expressed in terms of a slack variable.

A characteristic of all goal programming problems is that their criteria of optimality involve the minimization of one or more slack and/or surplus variables. This is because goal achievement is expressed in terms of either the amount of underachievement (value of the slack variable) or the amount of overachievement (value of the surplus variable).

In goal programming, each goal can be defined in one of three ways. These are:

1. Both under and overachievement are possible;
2. Only underachievement is possible; or
3. Only overachievement is possible.

Since it has been determined that 400,000 visitor days is the carrying capacity limit, the second approach where only underachievement is possible is used in the above example. That is, the above problem formulation does not permit the production of more than 400,000 visitor days. If both under and overachievement were to be permitted (approach 1), then both a slack variable (which measures underachievement) and a surplus variable (which measures overachievement) would need to be defined. If only overachievement is permitted, then only a surplus variable is defined.

### Defining the Second Goal

Now we shall consider the manager's second goal, that of maximizing timber production. Examination of the constraints for the original problem reveals that, unlike the case of the recreation goal, there is no constraint that reflects or incorporates the timber goal. If the constraint:

$$400x_1 + 100x_2 \leq 39,400,000$$

is added to the problem, then it is possible to reflect the timber goal.

This is done in the same fashion as with the recreation goal. For example, a slack variable, say  $x_4$ , is de-

defined such that

$$400x_1 + 100x_2 + x_4 = 39,400,000$$

This slack variable measures the difference between the actual amount of timber produced and the target value. That is:

$$x_4 = 39,400,000 - 400x_1 - 100x_2$$

As with recreation goal, we can state the timber goal in terms of the slack variable, which in this case is  $x_4$ . We know from the above discussion that we cannot produce more than 39,400,000 board feet of timber. Since increasing the amount of timber produced is the same as reducing the value of  $x_4$  (or vice versa), maximization of timber is achieved by minimizing  $x_4$ . The problem in terms of the timber goal is:

$$\min Z_2 = x_4$$

subject to:

$$\begin{array}{rcl} x_1 + x_2 & < & 100,000 \\ 400x_1 + 100x_2 & > & 20,000,000 \\ & 20x_2 & > & 40,000 \\ & 20x_2 & < & 400,000 \\ 400x_1 + 100x_2 + x_4 & = & 39,400,000 \\ x_1 \geq 0, x_2 \geq 0, x_4 \geq 0 \end{array}$$

Note that the timber constraint has been added to the problem.

### Dealing with Incommensurable Objectives

Up to now we have considered each goal separately. Since the goals are expressed in different units (board feet and visitor days), this is a situation where there are incommensurable objectives. Thus, simultaneous optimization of both of these goals as they are currently expressed is impossible. Two approaches to the solution of this problem that have been developed are cardinal weighting goal programming and pre-emptive ranking goal programming.

In the cardinal weighting approach, all goals are assumed to be in the same units and a single objective function is formulated. The coefficients in this objective function are, in effect, penalties for non-achievement of the various goals. In our example, suppose the objective function is:

$$\min Z = x_3 + x_4$$

In this case, both coefficients are 1 which means that the underachievement of the recreation goal has the same contribution per visit to  $Z$  as the underachievement of the timber goal does per board foot. In other words, the two goals have equal importance in this case. In a typical analysis using cardinal weighting, the coefficients or penalties must be differentially weighted in order to incorporate the decisionmaker's preferences. For example, a penalty of 1.5 might be attached to each visit by which the recreation goal is underachieved. The objective function (assuming the penalty for timber is unchanged) becomes:

$$\min Z = 1.5x_3 + x_4$$

This choice of penalties implies that the decisionmaker prefers underachievement of the timber goal (penalty of 1 per board foot) to the underachievement of the recreation goal (penalty of 1.5 per visit).

The problem with this approach lies in the choice of a single set of penalties or weights. For example, who is to say which of the sets of weights just given is better and why? Rather than doing this, a more rational approach is to consider several different sets of weights or penalties. This will result in the formulation of a number of objective functions and the problem can be solved for each one. Trade-offs between different levels of achievement of the goals can be determined and an analysis of these trade-offs can help the decisionmaker choose a preference structure for the goals being considered.

To properly apply this method, care must be taken in the choice of targets for each goal. An approach such as the one discussed above for the recreation and timber goals will insure that only non-inferior levels of goal achievement will be considered in the trade-off analysis. For any set of goals, a non-inferior set of achievement levels is defined to be one where there does not exist another feasible set of achievement levels, or where, for each goal, the actual achievement is at least as good as that for the set of achievement levels being considered. Said another way, inferior solutions lie in the interior of the feasible region rather than on the boundary.

For example, consider the following sets of achievement levels for the recreation and timber goals in Wilson's TR problem:

	Recreation	Timber
set 1	100,000	20,000,000
set 2	370,000	20,000,000
set 3	390,000	19,000,000
set 4	600,000	42,000,000

Clearly, set 2 is better than set 1 for recreation and equal to it for timber. Thus, set 1 is inferior to set 2 or, conversely, set 2 is non-inferior to set 1. Sets 2 and 3 are not comparable without additional information because one is better in recreation and the other is better in timber. (The same problem arises in a comparison of sets 1 and 3.) Set 4 is not feasible in terms of Wilson's TR problem, as we saw in the discussion of setting goal targets.

Setting targets, as was done for recreation, and then setting timber will ensure that sets like set 1 will not be considered in the trade-off analysis. Cardinal weighting goal programming is more efficient than LP for this type of analysis. For more detail on this approach, the reader is referred to the two papers by Field.

The pre-emptive ranking approach to goal programming considers each goal separately. In this approach, a priority or importance ranking is established and the achievement of the highest priority goal is considered first, after which the second priority goal is considered and so on for the remaining goals.

Suppose in our example that the recreation visitor day goal is assigned first priority and the timber goal second priority. In an attempt to satisfy the first priority goal, the following LP problem would be solved:

$$\min Z_1 = x_3$$

subject to:

$$\begin{array}{rcl} x_1 + x_2 & < & 100,000 \\ 400x_1 + 100x_2 & > & 20,000,000 \\ & 20x_2 & > & 40,000 \end{array}$$



$$\begin{aligned} 20x_2 + x_3 &= 400,000 \\ 400x_1 + 100x_2 + x_4 &= 39,400,000 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

Solution of this problem yields production of visitor days at the limit of feasibility or 400,000 as discussed above. In other words, this goal is completely achieved ( $x_3 = 0$ ) which is to be expected because of the way in which we chose the target.

Before attempting to satisfy the second priority goal, a constraint would be added to the problem to ensure that the level of achievement of the first priority goal is not violated. In this example, a suitable constraint would be  $x_3 = 0$ . In an attempt to satisfy the second priority goal, the following LP problem would be solved:

$$\begin{aligned} \min Z_2 &= x_4 \\ x_1 + x_2 &\leq 100,000 \\ 400x_1 + 100x_2 &\geq 20,000,000 \\ 20x_2 &\geq 40,000 \\ 20x_2 + x_3 &= 400,000 \\ 400x_1 + 100x_2 + x_4 &= 39,400,000 \\ x_3 &= 0 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 \end{aligned}$$

The optimal solution to this problem is the same as given above for the recreation goal. In each case, a total of 34,000,000 board feet are produced. The amount of timber produced does not increase even when the timber goal is considered because to increase timber production would require a reduction in recreation visits (see fig. 24), thus violating the constraint  $x_3 = 0$ . Note that, in this case, the LP formulation and the pre-emptive ranking goal formulation yield exactly the same optimal solution. This is not ordinarily the case. For example, if one reverses the goal priorities (i.e., making timber priority 1, etc.), the optimal solution will correspond to point D in figure 24. The reader may find it useful to verify this.

Solution of a pre-emptive goal programming problem consists of the solution of each of a sequence of LP problems, one for each goal priority level that has been defined. Each succeeding LP problem is modified by the addition of constraints to ensure that the level of achievement of all higher priority goals is not violated. We saw this in our example with the addition of the constraint  $x_3 = 0$  to the second LP problem to ensure that the level of achievement of visitor days not be reduced in attempting to satisfy the dollar benefits goals.

This characteristic of the pre-emptive ranking approach implies that goals at a given priority level are infinitely more important than those at lower priority levels. Suppose it were necessary to reduce the level of achievement of some goal by one unit in order to obtain one unit of a commodity associated with a lower priority goal. Since higher priority goal achievement cannot be violated, no units of the commodity associated with the lower priority goal would be produced. This implies that the last unit of the commodity associated with the higher priority goal is more important than the first unit of the commodity associated with the lower priority goal. Obviously, this is not very realistic since the trade-off between different commodities is rarely this clearcut in real world situations. A trade-off analysis using cardinal

weighting goal programming, as discussed above, is more likely to reflect preferences in such situations.

Another disadvantage of the pre-emptive ranking approach is that it is possible that one will end up with an inferior set of goal achievement levels. This is especially true if goal targets are selected in an arbitrary fashion, as has often been done in cases where this technique has been applied. As discussed above, this is also a problem with cardinal weighting goal programming. However, because of the characteristics of pre-emptive ranking, there is no way to guarantee that only non-inferior sets will be considered as can be done with the cardinal weighting approach. This is due, in part, to the fact that in the cardinal weighting approach there is only a single objective function and, hence, only a single pre-emptive rank. Said another way, no goal can pre-empt any other. For a good discussion of the disadvantages of pre-emptive ranking goal programming, the reader is referred to the paper by Dyer et al. (1979).

To summarize this brief introduction to goal programming, the following points are worth noting:

1. Both types of goal programming are special cases of LP and have all of the advantages and disadvantages associated with the latter technique.
2. In addition, both approaches to goal programming suffer from the disadvantages outlined above.
3. The objective functions for goal programming are always expressed in terms of slack and surplus variables. In goal programming terminology, these are called deviational variables because they measure the amount of deviation between actual goal achievement and the desired goal level. They are usually denoted by  $d^-$  for underachievements and  $d^+$  for overachievements.
4. In the pre-emptive ranking approach to goal programming, the final solution may be sub-optimal or inferior in that it lies on the interior of the feasible region. This is caused by the infinite weights described above for higher priority goals. Such a solution is sometimes called a *satisficing* solution.
5. Properly formulated cardinally-weighted goal programming can, more conveniently than LP, produce the trade-off surface in a multiple criterion problem and provide the basis for a compromise solution that is non-inferior.

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This report provides a detailed explanation of basic concepts of linear programming (LP) for forest planners and others who must use the technique to develop national forest land management plans. The role of LP in national forest planning analysis is presented. Basic algebraic concepts needed to understand LP are reviewed. The various components of an LP model are described and the advantages and limitations of the technique are discussed. Examples are presented in order to illustrate the technique, how it is used to formulate forest planning models, and how it is used in the Forest Service land management planning process. Procedures for solving LP models and the technique goal programming are also discussed. No prior background in operations research is assumed on the part of the reader.

**Keywords:** Management Planning, Linear Programming, Goal Programming, Multiple Use Management





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